Lecture 40

Dec 7 2011

159 Altgeld

Tutoring Fri & Mon Dec 12 10AM–2PM

Office Hours Fri 1–3.

Final exam cumulative. Covers everything from class, up to & including Lecture 38 (not Monday's material).

In the beginning...

Ch. 12 Vectors

Important topics

- vectors, vector addition
- dot product ⇝ norm or magnitude
  ⇝ orthogonal vectors
- cross product
  $\vec{u}\times\vec{v}$ orthog to both $\vec{u}$ & $\vec{v}$,
  $\|\vec{u}\times\vec{v}\|$ = area parallelogram
- Equation for plane: $\vec{P}$ point on plane, $\vec{n}$ normal vector
  $$\vec{n}\cdot(x,y,z) = \vec{n}\cdot\vec{P}$$
- Quadric surfaces (consider cross-sections)

ellipsoid $\quad \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} + \dfrac{z^2}{C^2} = 1$

(elliptic) paraboloid $\quad \pm z = Ax^2 + By^2$ $\quad$ both ≥ 0

(saddle) hyperbolic paraboloid $\quad \pm z = Ax^2 - By^2$

hyperboloid $\quad \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} - \dfrac{z^2}{C^2} = \pm 1$ $\quad$ one- or two-sheeted

Ch.13 Vector functions

- $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ parametrizes curve in $\mathbb{R}^3$
- $\vec{r}(t) = (x(t), y(t), z(t))$, $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

tangent vector to curve

Interpretation: $\vec{r}(t)$ = position, $\vec{r}\,'(t)$ = velocity,
$\|\vec{r}\,'(t)\|$ = speed

- Arc length $= \int \|r'(t)\|\,dt$ = ds

Ch. 14

$f(x,y,z) = c$

- level sets, contour maps
- limits for $f: \mathbb{R}^3 \to \mathbb{R}$ (ε, δ)
  - test along curves to show limit DNE
- continuity for $f: \mathbb{R}^3 \to \mathbb{R}$ (function value = limit value)
- partial derivatives (focus on one variable, treat others as constant)
- Clairaut's theorem $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$
- linear approx & tangent planes
  $P = (a,b)$. S = graph of $f(x,y)$.

  Tangent plane at P given by
  $$z - f(P) = f_x(P)(x-a) + f_y(P)(y-b)$$
  Linear approx near P
  $$L(x,y) = f(P) + f_x(P)(x-a) + f_y(P)(y-b)$$
- f differentiable @ P if $\lim_{(x,y)\to P} \frac{L(x,y) - f(x,y)}{\|(x,y) - P\|} = 0$.

- f diff. as long as f_x, f_y exist & continuous
- Chain Rule $\underset{u,v}{\mathbb{R}^2} \xrightarrow{g} \underset{x,y,z}{\mathbb{R}^3} \xrightarrow{f} \mathbb{R}$

$$\frac{\partial}{\partial u}(f \circ g) = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial g_3}{\partial u}$$

- Gradient $\nabla f = (f_x, f_y, f_z)$.

  Interpretation: • direction of maximal increase of f
  • orthogonal to level sets

  gives new way to find normal vector to surface:
  - write S as level set $f = c$, $\vec{n} = \nabla(f)$.

- Directional derivative: $D_{\vec{u}} f$ generalizes partial deriv

$$D_{\vec{u}} f = \nabla(f) \cdot \vec{u}$$

- critical points if f has local max/min at P, then $\nabla f(P) = \vec{0}$.
- 2^{nd} Der Test: $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$

  $\nabla f(P) = \vec{0}$

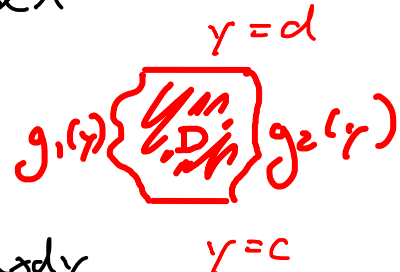
  If: • $D(P) < 0$ saddle point
  • $D(P) > 0$ & $\frac{\partial^2 f}{\partial x^2}(P) > 0$ local min
  • $D(P) > 0$ & $\frac{\partial^2 f}{\partial x^2}(P) < 0$ local max

- Extreme Value Thm: f continuous on closed & bounded domain Ω, then f has global max/min on Ω.
- to find global max/min of f on closed, bounded Ω, use 2^{nd} Der test on "interior" of Ω and look for max/min on boundary
  - If can write boundary as level set of g, then max/min on $\partial\Omega$ satisfy "Lagrange Multipliers"

$$\nabla f(P) = \lambda \nabla(g)(P)$$

Ch. 15 - double integral $\iint_D f\,dA$

- Can be computed as iterated integral



$$\iint_D f\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f\,dx\,dy$$

- "Fubini": can also be computed as $\iint f\,dy\,dx$ (may need to chop up D)
- interpretation: 1) average value of f on D $= \dfrac{1}{\text{area}(D)} \iint_D f\,dA$

  2) volume of shape w/ bottom D & height given by f
- triple integrals similar $\iiint_E f\,dV = \int_a^b \int_{f_1}^{f_2} \int_{g_1}^{g_2} f\,dx\,dy\,dz$

  "slice" region E
- change of coords: polar / cylindrical, spherical

  $\boxed{r\,dr\,d\theta}$ $\quad$ $\boxed{\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta}$

- General change-of-coords

  $T(u,v,w) = (x,y,z)$

  Jacobian $J(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

  Then if $E = T(R)$, get

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_R f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J(T)|\,du\,dv\,dw$$

Ch. 16 - vector fields $\vec{F}$, $\vec{F}$ conservative if $\vec{F} = \nabla f$

- line integrals of functions: $\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

- interpretation: area of curved region
- line integrals of vector fields $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

  interpretation: work
- Fundamental Theorem $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$
- 1st Recognition Theorem

  $\vec{F}$ conservative $\iff$ $\int_C \vec{F} \cdot d\vec{r}$ is path independent
- 2nd Recognition Theorem

  $\vec{F}$ conservative on simply-conn domain $\iff$ $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
- Green's thm: $\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dA$
- curl $\vec{F} = \nabla \times \vec{F}$

  interpretation: points in direction of rotational axis

  2nd Recognition theorem for $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

  $\vec{F}$ conservative on simply-conn domain $\iff \nabla \times \vec{F} = \vec{0}$
- div $\vec{F} = \nabla \cdot \vec{F}$

  interpretation: measures rate of expansion/compression

  - for $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = \nabla \times \vec{G} \iff \operatorname{div} \vec{F} = 0$.
- $\vec{r}(u,v)$ parametrizing S, then normal vector

  $\vec{n} = \vec{r}_u \times \vec{r}_v$
- surface integral of function $\iint_S f\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$

- average value of f on S $= \dfrac{1}{\text{area}(S)} \iint_S f \, dS$.

- surface integral of vector field $\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \underbrace{\vec{n}}_{\vec{r}_u \times \vec{r}_v} \, dS$

  interpretation: flux

- Stokes' thm $\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot dS$

  (generalizes Green's thm)

- Divergence thm either $\int_{\partial S} \vec{F} \cdot \vec{n} \, ds = \iint_S \operatorname{div} \vec{F} \, dS$

  or $\iint_{\partial E} \vec{F} \cdot \vec{n} \, dS = \iiint_E \operatorname{div} \vec{F} \, dV$