Lecture 5

Aug. 31
2011

§14.1: Functions of several variables (single-valued)

How do we think about functions?

Several ways:

1. via a formula, like $f(x) = x^2 + 1$

2. a mysterious "black box"

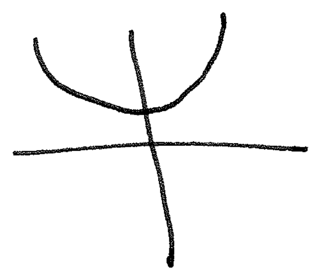
put something in the box, get something else out

Domain allowable inputs

Range possible outputs

Example: Dow Jones industrial average, as a function of time.

3. via a graph: the collection of all (x, y) such that $y = f(x)$



Also true for functions of several variables.

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f(x,y) = x^2 + y^2$

domain D is $\mathbb{R}^2$, range R is $\{ z \mid z \geq 0 \}$

What does the graph of f look like?

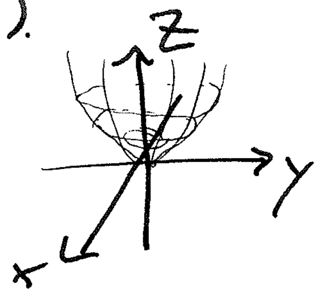
For $g: \mathbb{R} \to \mathbb{R}$, the graph of g is contained in $\mathbb{R}^2$. One coordinate for input, one for output

For $f: \mathbb{R}^2 \to \mathbb{R}$, graph should now be in $\underline{\underline{\mathbb{R}^3}}$, since have two input coordinates.

More generally, if $F: \mathbb{R}^n \to \mathbb{R}$, then the graph of F is in $\mathbb{R}^{n+1}$.

Graph of f is $\{(x, y, z) \mid z = f(x,y)\}$

Since f is function of x & y, for each pair (x,y), there is a single z for which (x, y, z) is in the graph. (Like the "vertical line test" for functions $\mathbb{R} \to \mathbb{R}$).



In general, may be difficult to understand the graph of $f: \mathbb{R}^2 \to \mathbb{R}$. What are some techniques?

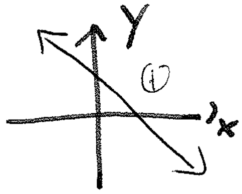
1) Intersect the graph w/ planes.

<u>Example</u> $f(x,y) = 1-x-y$.

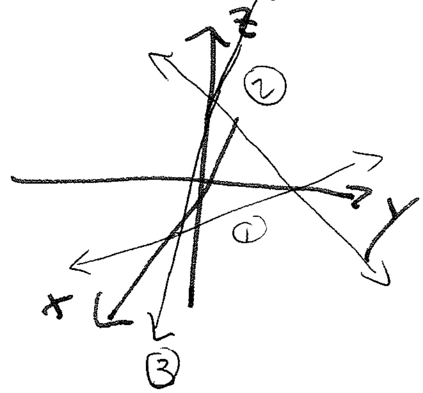
Intersect w/ $z=0$ plane.

Solving for $1-x-y=0$. Find $x+y=1$, or $y = 1-x$

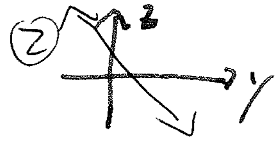
The graph of $y = 1 - x$ is



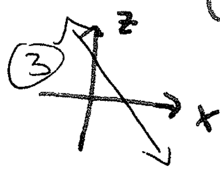
Rotate



Intersect w/ $x = 0$ plane, get $z = 1 - y$.

In the (y, z) plane, this is 

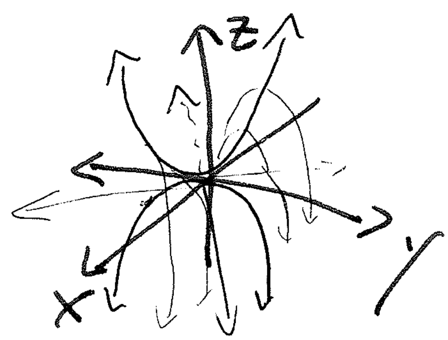
Finally, intersecting w/ $y = 0$ plane gives $z = 1 - x$.

Get 

in (x, z) plane

This is the plane $x + y + z = 1$.

<u>Example</u> $f(x, y) = x^2 - y^2$.



Intersect w/ $x = 0$ plane,

get $z = -y^2$

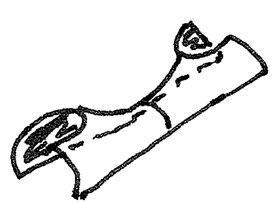
Intersect w/ $y = 0$ plane,

get $z = x^2$

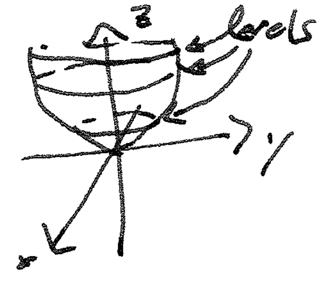
Intersect w/ $z = 0$ plane, get $x^2 - y^2 = 0$, or $x^2 = y^2$.

If $x^2 = y^2$, then $x = \pm y$, so get the <u>union</u> of the lines $x = y$, $x = -y$.

Resulting picture is a "saddle"



Often interesting to intersect graph with horizontal planes $z=c$. These intersections are called "level sets" or "level curves".



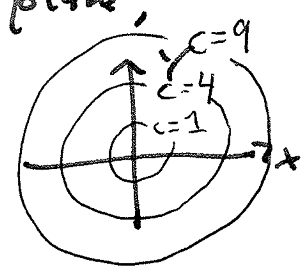
Example $f(x,y)=x^2+y^2$.

The level sets with $c<0$ are empty.

For $c=0$, get a point.

For $c>0$, get a circle (radius $\sqrt{c}$).

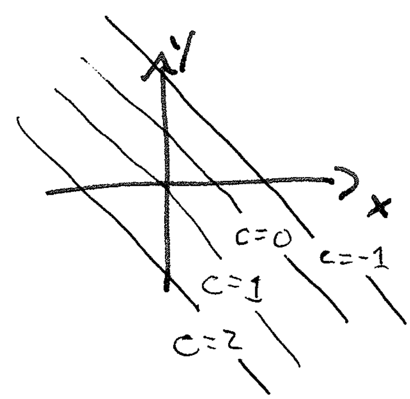
If we overlay all level sets on the (x,y) plane, get a "contour map" of the function.



Example $f(x,y)=1-x-y$.

Level set $1-x-y=c \iff y=1-c-x$

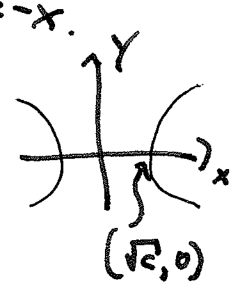
All lines of slope -1, contour map is



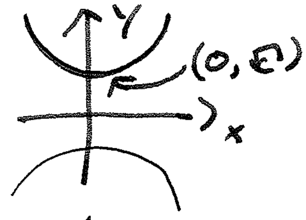
Example $f(x,y)=x^2-y^2$

Saw $c=0$ is lines $y=x$ & $y=-x$.

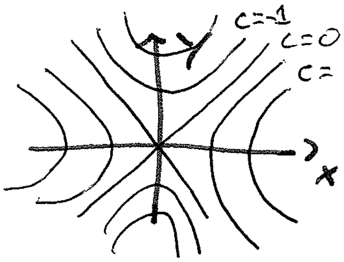
For $c>0$, $c=x^2-y^2$ get



For $c<0$, $c=x^2-y^2$ get



So the contour map looks like



This becomes more difficult for functions of <u>more than</u> two variables. (Can't draw).

Can again try "level sets" approach to reduce dimension.

The intersection of the graph of $f(x,y,z)$ w/ the "hyperplane" $c=0$ will give a 2-dimensional object (a "surface"). So may be able to draw <u>level surfaces</u>.

<u>Example</u> $f(x,y,z)=x^2+y^2+z^2$.

Level surfaces for $c<0$ are empty.

For $c=0$, have a point.

For $c>0$, have sphere of radius $\sqrt{c}$.

The "contour map" is now in $\mathbb{R}^3$.