Lecture 7

Sept. 7
2011

Last time: Limits for $f: \mathbb{R} \to \mathbb{R}$.

Rough definition: The limit of f as $x \to c$ exists & is equal to L if we can force $f(x)$ to be as close as we want to L by picking x sufficienty near c.

Precise definition: The limit of f as $x \to c$ exists & is equal to L if:

$$\boxed{\begin{array}{l}\text{for every } \varepsilon \text{ (epsilon)} > 0 \text{ there is } \delta \text{ (delta)} > 0 \\ \text{such that if } 0 < |x-c| < \delta \text{ then } |f(x) - L| < \varepsilon.\end{array}}$$

Now let $f: \mathbb{R}^n \to \mathbb{R}$. (Will discuss $f: \mathbb{R}^3 \to \mathbb{R}^3$ in Chapter 16)

To make sense of $\displaystyle\lim_{(x_1, x_2, \ldots, x_n) \to (c_1, c_2, \ldots, c_n)} f(x_1, \ldots, x_n) = L$,

what must be changed. Just replace $|x-c|$ by $\|(x_1, \ldots, x_n) - (c_1, \ldots, c_n)\|$

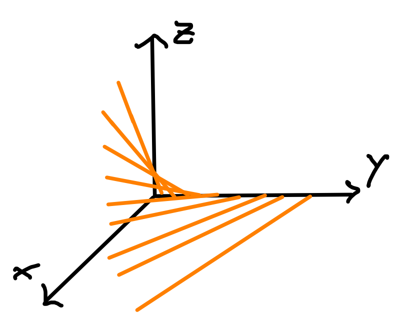
So, for $f: \mathbb{R}^n \to \mathbb{R}$, the limit of f as $(x_1, \ldots, x_n) \to (c_1, \ldots, c_n)$ exists and is equal to L if:

$$\boxed{\begin{array}{l}\text{for every } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ \text{if } 0 < \|(x_1, \ldots, x_n) - (c_1, \ldots, c_n)\| < \delta \text{ then} \\ |f(x_1, \ldots, x_n) - L| < \varepsilon.\end{array}}$$

Example: 1) $f(x,y) = \dfrac{x}{y}$.

Does $\displaystyle\lim_{(x,y) \to (1,1)} f(x,y)$ exist?

What about $\displaystyle\lim_{(x,y) \to (1,0)} f(x,y)$?



At $(1,1)$: $f(1,1)=1$ and expect $\lim_{(x,y)\to(1,1)} f(x,y) = 1$. Let's show this.

Let $\varepsilon > 0$. We need to find a $\delta > 0$ that works for this particular ε. Want to make $\left|\frac{x}{y} - 1\right| < \varepsilon$ by making $0 < \|(x,y) - (1,1)\| < \delta$

$$\|(x,y)-(1,1)\| = \sqrt{(x-1)^2 + (y-1)^2}.$$

If we need to make $\left|\frac{x}{y} - 1\right| = \frac{|x-y|}{|y|}$ small, can try to make both $|x-y|$ and $\frac{1}{|y|}$ small.

First deal with $\frac{1}{|y|}$. Since y is near 1, $\frac{1}{|y|}$ will be close to 1. In particular, if we take $\delta < \frac{1}{2}$ then $|y-1| \le \sqrt{(x-1)^2 + (y-1)^2} < \delta < \frac{1}{2}$, so

$\frac{1}{2} < y < \frac{3}{2}$. So $\frac{1}{|y|} = \frac{1}{y} < 2$.

Then $\frac{|x-y|}{|y|} < 2|x-y|$.

So in order to get $\frac{|x-y|}{|y|} < \varepsilon$, enough to make $|x-y| < \varepsilon/2$. Use the triangle inequality:

$|x-y| \le |x-1| + |1-y|$, so enough to make

$|x-1| \le \varepsilon/4$ and $|1-y| \le \varepsilon/4$.

But both and are $\le \sqrt{(x-1)^2 + (y-1)^2} < \delta$, so we just need to take $\delta < \varepsilon/4$. OK, now apply the two constraints: $\delta < \frac{1}{2}$ and $\delta < \varepsilon/4$.

Take any δ smaller than both $\frac{1}{2}$ and $\varepsilon/4$.

Then, if $0 < \|(x,y) - (1,1)\| < \delta$, it will be true that $|f(x,y) - 1| < \varepsilon$. So $\lim_{(x,y)\to(1,1)} f(x,y) = 1$

At $(1,0)$: $f(1,0)$ is not defined, but that does not mean $\lim_{(x,y)\to(1,0)} f(x,y)$ cannot exist.

In fact, the limit does not exist. We could do this from scratch, but there is an easier way.

Key: If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is L, then the limit must also exist and be equal to L if (x,y) approaches (a,b) along a line (or a curve).

So, we can consider $(x,y)\to(1,0)$ on the line $x=1$. The limit along this line is $\lim_{y\to 0} \frac{1}{y}$, which DNE (from Calc I). So $\lim_{(x,y)\to(1,0)} f(x,y)$ also DNE.

Example: Find $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$.

Do we think the limit exists? Approach along x-axis & y-axis. Along x:

$$\lim_{(x,0)\to(0,0)} \frac{x}{x} = 1$$

Along y:

$$\lim_{(0,y)\to(0,0)} \frac{-y}{y} = -1.$$

So $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ DNE.

Note: Can get any value as a limit along a line. $M = \frac{1-L}{1+L}$

To get a limit of L, approach along the line $y = \frac{1-L}{1+L}x$

$$\lim_{(x,Mx)\to(0,0)} \frac{x-Mx}{x+Mx} = \lim_{x\to 0} \frac{1-M}{1+M} = \lim_{x\to 0} \frac{2L/1+L}{2/1+L} = L.$$

Example: Find $\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2-3y^6}$

Try along line $y=mx$.

$$\lim_{(x,mx)\to(0,0)} \frac{2x(mx)^3}{x^2-3(mx)^6} = \lim_{(x,mx)\to(0,0)} \frac{2m^3x^4}{x^2-3m^6x^6}$$

$$= \lim_{(x,mx)\to(0,0)} \frac{2m^3x^2}{1-3m^6x^4} = 0.$$

Great! Maybe $\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2-3y^6} = 0$. (Get same answer along line $x=0$).

Actually, <u>NO</u>. Approach along curve $x=y^3$. Get

$$\lim_{(y^3,y)\to(0,0)} \frac{2y^3\cdot y^3}{y^6-3y^6} = \frac{2y^6}{-2y^6} = -1. \text{ So}$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2-3y^6} \quad \text{DNE.}$$

So how to tell when limits do exist?

1) If $f(x,y)$ is a polynomial in x & y, then
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

2) If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ & $\lim_{(x,y)\to(a,b)} g(x,y) = M$,

then $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L+M$

and $\lim_{(x,y)\to(a,b)} [f(x,y)\cdot g(x,y)] = L\cdot M$.

If $M \neq 0$, then
$$\lim_{(x,y)\to(a,b)} [f(x,y)/g(x,y)] = L/M$$