Lecture 8

Sept. 9 2011 ①

Last time Limits for $f: \mathbb{R}^n \to \mathbb{R}$.

To understand $\lim_{\vec{x} \to \vec{c}} f(\vec{x})$, approach $\vec{c}$ along lines or curves.

If 1) some of these are different, then

$$\lim_{\vec{x} \to \vec{c}} f(\vec{x}) \text{ DNE}$$

OR 2) all of these limits agree, then probably $\lim_{\vec{x} \to \vec{c}} f(x)$ is same.

Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Along any line, find limit of 0

$$\left(\text{Along line } y = mx, \text{ get } \lim_{x\to 0} \frac{mx^2}{|x|(1+m^2)} = \lim_{x \to 0} \frac{m}{1+m^2}|x| = 0 \right)$$

Also along parabolas. How to show it is 0?

Use the Squeeze Theorem:

If $f \le g \le h$ and $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$,

then $\lim_{(x,y)\to(a,b)} g(x,y) = L$ too.

Enough (by Squeeze thm) to show

$$\lim \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = 0.$$

But $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$, so

$$0 \leq |x| \cdot \frac{|y|}{\sqrt{x^2+y^2}} \leq |x|.$$

Since $\lim\limits_{(x,y)\to(0,0)} |x| = 0$, done by Squeeze Theorem.

Now understand limits, so can discuss continuity.

Definition $f: \mathbb{R}^n \to \mathbb{R}$ is continuous at $\vec{c}$ if

$$\lim_{\vec{x}\to\vec{c}} f(\vec{x}) = f(\vec{c}).$$

Examples
- polynomial functions $f(\vec{x}) = p(\vec{x})$ (continuous everywhere)
- rational functions

$$g(\vec{x}) = \frac{p(\vec{x})}{q(\vec{x})}$$ polynomial functions

continuous wherever $q(\vec{x}) \neq 0$.

- $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$

Easy to see continuous away from origin.

At $(x,y) = (0,0)$, f is not defined.

Define $g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

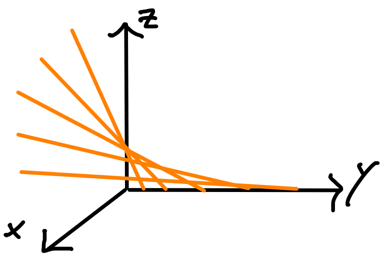
Then g is continuous everywhere since

$$\lim_{(x,y)\to(0,0)} g = \lim_{(x,y)\to(0,0)} f = 0 \quad \text{(above example)}$$

§14.3 (Partial) Derivatives

For $f:\mathbb{R}\to\mathbb{R}$, $f'(c) = \frac{df}{dx}(c)$ measures rate of change of $f(x)$ at $x=c$. (Change as $x\to c$).

For $f:\mathbb{R}^n\to\mathbb{R}$ f may change differently depending on direction of approach.



Example $f(x,y) = \frac{x}{y}$

At $(1,1)$, f increasing as x increases, decreasing as y increases.

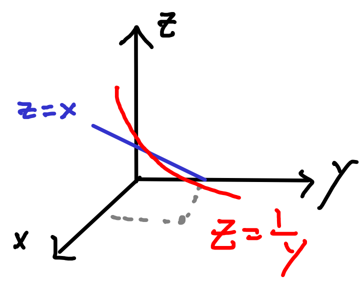
Rate of change in x direction is partial derivative with respect to x, written $\frac{\partial f}{\partial x}(x,y)$ or $f_x(x,y)$.

Definition: Let $g(x) = f(x,d)$ (for fixed d).

Then $\frac{\partial f}{\partial x}(c,d) = g'(c)$.

Similarly, let $h(y) = f(c,y)$ (fixed c)

Then $\frac{\partial f}{\partial y}(c,d) = h'(d)$.



Example $f(x,y) = \frac{x}{y}$, at $(1,1)$.

$$\frac{\partial f}{\partial x}(1,1) = \frac{d}{dx}(x)\Big|_{x=1} = 1.$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{d}{dy}\left(\frac{1}{y}\right)\Big|_{y=1} = \left(\frac{-1}{y^2}\right)\Big|_{y=1} = -1$$

Expanded definition:

Recall that $g'(c)$ means $\lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$

or $\lim_{x\to c} \frac{g(x)-g(c)}{x-c}$

So really $\dfrac{\partial f}{\partial x}(c,d) = \lim\limits_{h \to 0} \dfrac{f(c+h,d) - f(c,d)}{h}$

OR $\quad = \lim\limits_{x \to c} \dfrac{f(x,d) - f(c,d)}{x - c}$

and $\quad \dfrac{\partial f}{\partial y}(c,d) = \lim\limits_{h \to 0} \dfrac{f(c,d+h) - f(c,d)}{h}$

$$= \lim_{y \to d} \frac{f(c,y) - f(c,d)}{y - d}$$

For $f(x,y) = \frac{x}{y}$,

$$f_x(c,d) = \lim_{h \to 0} \frac{\frac{c+h}{d} - \frac{c}{d}}{h} = \lim_{h \to 0} \frac{h}{d \cdot h} = \frac{1}{d}.$$

and $f_y(c,d) = \lim\limits_{h \to 0} \dfrac{\frac{c}{d+h} - \frac{c}{d}}{h} = \lim\limits_{h \to 0} \dfrac{\frac{c \cdot d - c \cdot (d+h)}{(d+h)d}}{h}$

$$= \lim_{h \to 0} \frac{-c \cdot \cancel{h}}{(d+h) \cdot d \cdot \cancel{h}} = \lim_{h \to 0} \frac{-c}{(d+h)d} = \frac{-c}{d^2}.$$

Even more notation:

$$\frac{\partial f}{\partial x}(c,d) = f_1(c,d) = D_1 f(c,d) = D_x f(c,d)$$

and $\dfrac{\partial f}{\partial y}(c,d) = f_2(c,d) = D_2 f(c,d) = D_y f(c,d).$

For more than 2 variables, all works the same way

Ex $\quad f(x,y,z,w) = 2x^3yw - 6y^2zw + z^2w^2$

$$f_1 = f_x = \frac{\partial f}{\partial x} = 6x^2yw$$

$$f_2 = f_y = \frac{\partial f}{\partial y} = 2x^3w - 12yzw$$

$$f_3 = f_z = \frac{\partial f}{\partial z} = 6y^2w + 2zw^2$$

$f_y = f_w = \frac{\partial f}{\partial w} = 2x^3y - 6y^2z + 2z^2w$

<u>Higher derivatives</u> $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc. are still functions $\mathbb{R}^n \to \mathbb{R}$, so can consider <u>their</u> partial derivatives.

<u>Ex</u> Same f as above

$f_{y,x} = D_2 D_1 f = 6x^2w$

$f_{x,x} = D_1 D_1 f = 12xyw$ same

$f_{x,y} = D_1 D_2 f = 6x^2w$

$f_{w,x} = D_4 D_1 f = 6x^2y$

same

$f_{x,w} = D_1 D_4 f = 6x^2y$

In fact, for any f, if f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. (This is Clairaut's Theorem)

Alternative notation for higher partial derivatives:

$$f_{x,y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{x,x} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial^2 f}{\partial x^2}$$

<u>Careful</u>: This is just notation. Do not try to interpret it literally.