Lecture 9

Sept. 12
2011

Last time: Partial derivatives

$$f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)$$ partial derivative with respect to x.

Treat all other variables as constants, take derivative in x as usual.

Ex $f(x,y,z,w) = 2x^3yw - 6y^2zw + z^2w^2$

$$f_1 = f_x = \frac{\partial f}{\partial x} = 6x^2yw$$

$$f_2 = f_y = \frac{\partial f}{\partial y} = 2x^3w - 12yzw$$

$$f_3 = f_z = \frac{\partial f}{\partial z} = 6y^2w + 2zw^2$$

$$f_4 = f_w = \frac{\partial f}{\partial w} = 2x^3y - 6y^2z + 2z^2w$$

Higher derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc. are still functions $\mathbb{R}^n \to \mathbb{R}$, so can consider their partial derivatives.

Ex Same f as above

$$(f_x)_y = f_{x,y} = D_2 D_1 f = 6x^2w$$

$$(f_x)_x = f_{x,x} = D_1 D_1 f = 12xyw$$

$$(f_y)_x = f_{y,x} = D_1 D_2 f = 6x^2w$$

same

$$f_{w,x} = D_4 D_1 f = 6x^2y$$

$$f_{x,w} = D_1 D_4 f = 6x^2y$$

same

In fact, for any f, if f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. (This is Clairaut's Theorem)

Alternative notation for higher partial derivatives:

$$f_{x,y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{x,x} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f = \frac{\partial^2 f}{\partial x^2}$$

Careful: This is just notation. Do not try to interpret it literally.

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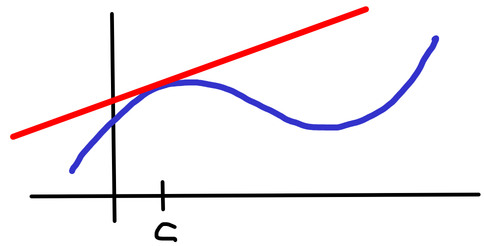
§ 14.4 Tangent Planes & Linear Approximations.

What are derivatives good for?

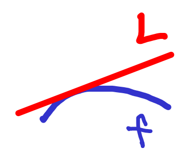
For $f: \mathbb{R} \to \mathbb{R}$,

$f'(c) =$ slope of tangent line at $x = c$



The tangent line is given by $y = f'(c)(x-c) + f(c)$.

The function $L(x) = f'(c)(x-c) + f(c)$ is "best linear approximation to f at $x = c$".



For x near c, L near f.

$$\lim_{x \to c} \frac{f(x) - L(x)}{x - c} = 0.$$

Why? Expand:

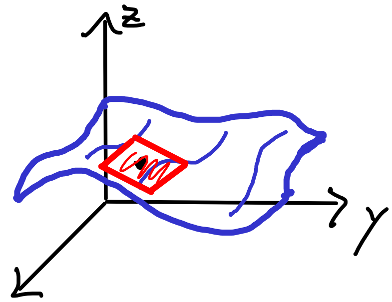
$$\lim_{x\to c} \frac{f(x) - L(x)}{x-c} = \lim_{x \to c} \frac{f(x) - [f'(c)(x-c) + f(c)]}{x-c}$$

$$= \lim_{x\to c} \frac{f(x) - f(c)}{x-c} - f'(c) = 0$$

by definition of $f'(c)$.

Try to do this for $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Instead of tangent line, want <u>tangent</u> <u>plane</u>.



The "best linear approximation" is now $L(x,y) = f(c,d) + \frac{\partial f}{\partial x}(c,d)(x-c) + \frac{\partial f}{\partial y}(c,d)(y-d)$.

<u>Equation of plane</u>: $z - f(c,d) = \frac{\partial f}{\partial x}(c,d)(x-c) + \frac{\partial f}{\partial y}(c,d)(y-d)$.

Specified by point on plane $(c, d, f(c,d))$

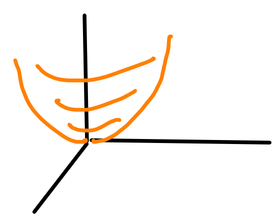
and normal vector $\left(\frac{\partial f}{\partial x}(c,d), \frac{\partial f}{\partial y}(c,d), -1\right)$

When is this a good approximation?

$$\lim_{(x,y)\to(c,d)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (c,d)\|} = 0.$$

<u>Definition</u>: $f: \mathbb{R}^2 \to \mathbb{R}$ is <u>differentiable</u> at c,d if the above limit is zero.

<u>Example</u> $f(x,y) = x^2 + y^2$ (elliptic paraboloid)



Is this diff. at $(1,1)$?

$f_x(x,y) = 2x \qquad f_x(1,1) = 2$

$f_y(1,1) = 2y \qquad f_y(1,1) = 2,$

$L(x,y) = 2 + 2(x-1) + 2(y-1).$

Then
$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2 - [2 + 2(x-1) + 2(y-1)]}{\sqrt{(x-1)^2 + (y-1)^2}} = \lim_{(x,y)\to(1,1)} \frac{(x-1)^2 + (y-1)^2}{\sqrt{(x-1)^2 + (y-1)^2}}$$

$$= \lim_{(x,y)\to(1,1)} \sqrt{(x-1)^2 + (y-1)^2} = 0. \quad \text{YES.}$$

Example $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Is this diff. at $(0,0)$?

Find f_x, f_y.

$$f_x(x,y) = \frac{y(x^2+y^2) - (xy)2x}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$ not defined at $(0,0)$.

What is $f_x(0,0)$? Back to defn

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - 0}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$ So $f_x(0,0) = 0$.

Similarly, $f_y(x,y) = \dfrac{x^3 - xy^2}{(x^2+y^2)^2}$ and $f_y(0,0) = 0$.

So $L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(y-0) = \underline{\underline{0}}$.

Is 0 a good approx to f near $(0,0)$?

Will answer this next time.