Math 241 - Solutions to Quiz 6- Thursday, December 1

1. Let C be the semi-ellipse $x^2 + 4y^2 = 1$, $x \geq 0$ and consider the vector field

$$F(x,y) = (1, x^2).$$

(a) Find a parametrization $\mathbf{r}(t)$ of C and normal vector $\mathbf{n}(t)$ pointing to the outside of the ellipse. (3 points)

SOLUTION:

$$\mathbf{r}(t) = (\cos t, 1/2 \sin t),\ -\pi/2 \leq t \leq \pi/2$$

$$\mathbf{r}'(t) = (-\sin t, 1/2 \cos t),\ -\pi/2 \leq t \leq \pi/2$$

$$\mathbf{n}(t) = (1/2 \cos t, \sin t),\ -\pi/2 \leq t \leq \pi/2$$

We see that $\mathbf{n}(t)$ is normal to the curve since $\mathbf{r}'(t) \cdot \mathbf{n}(t) = 0$ for every t. It points outward since it is never 0 and $\mathbf{n}(0) = (1/2, 0)$ points outward.

(b) Find the flux of F across C, moving from inside the ellipse to outside the ellipse. If you couldn't do (a), use $\mathbf{r}(t) = (\cos t, \sin t)$ and $\mathbf{n}(t) = (\cos t, \sin t)$. (3 points)

SOLUTION:

Flux of F across C from inside to outside $= \int_C F \cdot \mathbf{u} ds$, where $\mathbf{u}$ is an outward pointing unit normal. For the choice of $\mathbf{n}$ in (a), $\mathbf{u} = \mathbf{n}/|\mathbf{n}|$ and $|\mathbf{n}| = |\mathbf{r}'|$ so

$$\int_C (1, x^2) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} ds = \int_0^{2\pi} (1, \cos^2 t) \cdot \frac{(1/2 \cos t, \sin t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| dt$$

$$= \int_{-\pi/2}^{\pi/2} 1/2 \cos t + \cos^2 t \sin t dt = [1/2 \sin t - 1/3 \cos^3 t]_{-\pi/2}^{\pi/2} = 1/2 - (-1/2) = 1$$

2. Let $F(x,y) = (x^3, -x^2y)$.

(a) Find div F. (2 points)

SOLUTION:

$$\operatorname{div} F = 3x^2 - x^2 = 2x^2$$

(b) Use the divergence theorem to compute the flux of F across the unit circle (with outward pointing normal vector).

SOLUTION:

The divergence theorem for curves (another form of Green's Theorem) states

$$\int_C F \cdot \mathbf{n} ds = \int \int_D \operatorname{div} F dA,$$

where C is the boundary curve of D. So if C is the unit circle and D is the unit disk we have

$$\int_C (x^3, -x^2y) \cdot \mathbf{n} ds = \int \int_D 2x^2 dA = \int_0^1 \int_0^{2\pi} 2r^2 \cos^2 \theta r dr d\theta$$

$$= \left(\int_0^1 2r^3 dr \right) \left(\int_0^{2\pi} \cos^2 \theta d\theta \right) = \frac{1}{2}\pi$$