Math 241 - Quiz 2 - Tuesday, October 4

Version 1

Your name here:

1. Let $f(x,y) = \sin x \sin y$.

   (a) Find $\nabla f(x,y)$. (1 point)

   SOLUTION:

   $\nabla f(x,y) = \langle \cos x \sin y, \sin x \cos y \rangle$

   (b) Find all critical points of f and <u>use the second derivative test</u> to identify their types. (5 points)

   SOLUTION:

   The system $\cos x \sin y = 0$ and $\sin x \cos y = 0$ is solved if either $\cos x = \cos y = 0$ or $\sin y = \sin x = 0$. Notice that we cannot have $\cos x = 0$ and $\sin x = 0$ since $\cos^2 x + \sin^2 y = 1$. We have:

$$\cos x = \cos y = 0 \Rightarrow \begin{cases} x = \frac{\pi}{2} + n\pi & n \text{ is any integer} \\ y = \frac{\pi}{2} + m\pi & m \text{ is any integer} \end{cases} \tag{1}$$

   or

$$\sin y = \sin x = 0 \Rightarrow \begin{cases} x = n\pi & n \text{ is any integer} \\ y = m\pi & m \text{ is any integer} \end{cases} \tag{2}$$

   To use the second derivative test, we compute $f_{xx} = -\sin x \sin y$, $f_{yy} = -\sin x \sin y$, and $f_{xy} = \cos x \cos y$. Call the Hessian D. We then have $D = f_{xx} f_{yy} - (f_{xy})^2 = (\sin x \sin y)^2 - (\cos x \cos y)^2$. In case (2) above $D = -(\cos x \cos y)^2 = -1 < 0$ (remember we have $\cos^2 x + \sin^2 x = 1$ so if $\sin x = 0$ then $\cos^2 x = 1$). So all the points in (2) are saddle points. In case (1) above $D = (\sin x \sin y)^2 = 1 > 0$, so we then analyze $f_{xx} = -\sin x \sin y$ for $x = \frac{\pi}{2} + n\pi$ and $y = \frac{\pi}{2} + m\pi$. Notice that $\sin(\pi/2 + n\pi)$ is 1 if n is even and -1 if n is odd. This can be stated compactly by writing $\sin(\pi/2 + n\pi) = (-1)^n$. We have

$$\begin{aligned} f_{xx}(\pi/2 + n\pi, \pi/2 + m\pi) &= \sin(\pi/2 + n\pi)\cos(\pi/2 + m\pi) \\ &= -(-1)^n(-1)^m \\ &= (-1)^{m+n+1} \end{aligned}$$

   Hence $f_{xx} > 0$ when $m + n$ is odd and $f_{xx} < 0$ when $m + n$ is even. The table below summarizes these findings:

Point	min / max / saddle
$(\pi/2+n\pi, \pi/2+m\pi)$ with $m+n$ odd	min
$(\pi/2+n\pi, \pi/2+m\pi)$ with $m+n$ even	max
$(n\pi, m\pi)$	saddle

For instance, if we restrict to the box $-\pi/2 \leq x \leq \pi/2$ and $-\pi/2 \leq y \leq \pi/2$ then we have 5 critical points: $(-\pi/2, -\pi/2)$ and $(\pi/2, \pi/2)$ are both maximums, $(-\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$ are both minimums, and $(0,0)$ is a saddle point.

(**OVER**)

2. Use the method of Lagrange multipliers to find the closest point to $P = (-3, 1)$ on the line $4x - 3y = 5$. (4 points)

SOLUTION:

Minimize $D = (x+3)^2 + (y-1)^2$, the function measuring distance from P to (x, y), subject to the restraint $g = 0$ where $g = 4x - 3y - 5$. We have $\nabla D = \langle 2(x+3), 2(y-2) \rangle$ and $\nabla g = \langle 4, -3 \rangle$. Using the method of Lagrange multipliers we get the system of equations:

$$\begin{aligned} (x+3) &= 2\lambda \\ 2(y-2) &= -3\lambda \end{aligned}$$

Solving for x and y in terms of λ we get $x = 2\lambda - 3$ and $y = -3/2\lambda + 2$. Plugging these values for x and y into $4x - 3y - 5 = 0$ we obtain that $\lambda = 8/5$ and so $x = 1/5$ and $y = -7/5$.

Math 241 - Quiz 2 - Tuesday, October 4

Version 2

Your name here:

1. Let $f(x, y) = \cos x \cos y$.

   (a) Find $\nabla f(x, y)$. (1 point)

   SOLUTION:

   $\nabla f(x, y) = \langle -\sin x \cos y, -\cos x \sin y \rangle$

   (b) Find all critical points of f and <u>use the second derivative test</u> to identify their types. (5 points)

   SOLUTION:

   The system $-\sin x \cos y = 0$ and $-\cos x \sin y = 0$ is solved if either $\cos x = \cos y = 0$ or $\sin y = \sin x = 0$. Notice that we cannot have $\cos x = 0$ and $\sin x = 0$ since $\cos^2 x + \sin^2 y = 1$. We have:

$$\cos x = \cos y = 0 \Rightarrow \begin{cases} x = \frac{\pi}{2} + n\pi & n \text{ is any integer} \\ y = \frac{\pi}{2} + m\pi & m \text{ is any integer} \end{cases} \tag{1}$$

or

$$\sin y = \sin x = 0 \Rightarrow \begin{cases} x = n\pi & n \text{ is any integer} \\ y = m\pi & m \text{ is any integer} \end{cases} \tag{2}$$

To use the second derivative test, we compute $f_{xx} = -\cos x \cos y$, $f_{yy} = -\cos x \cos y$, and $f_{xy} = \sin x \sin y$. Call the Hessian D. We then have $D = f_{xx} f_{yy} - (f_{xy})^2 = (\cos x \cos y)^2 - (\sin x \sin y)^2$. In case (1) above $D = -(\sin x \sin y)^2 = -1 < 0$ (remember we have $\cos^2 x + \sin^2 x = 1$ so if $\cos x = 0$ then $\sin^2 x = 1$). So all the points in (1) are saddle points. In case (2) above $D = (\cos x \cos y)^2 = 1 > 0$, so we then analyze $f_{xx} = -\cos x \cos y$ for $x = n\pi$ and $y = m\pi$. Notice that $\cos(n\pi)$ is 1 if n is even and -1 if n is odd. This can be stated compactly by writing $\cos(n\pi) = (-1)^n$. We have

$$\begin{aligned} f_{xx}(n\pi, m\pi) &= -\cos(n\pi)\cos(m\pi) \\ &= -(-1)^n(-1)^m \\ &= (-1)^{m+n+1} \end{aligned}$$

Hence $f_{xx} > 0$ when $m + n$ is odd and $f_{xx} < 0$ when $m + n$ is even. The table below summarizes these findings:

Point	min / max / saddle
$(n\pi, m\pi)$ with $m+n$ odd	min
$(n\pi, m\pi)$ with $m+n$ even	max
$(\pi/2+n\pi, \pi/2+m\pi)$	saddle

For instance, if we restrict to the box $0 \leq x\pi$ and $0 \leq y \leq \pi$ then we have 5 critical points: $(0,0)$ and (π,π) are both maximums, $(0,\pi)$ and $(\pi,0)$ are both minimums, and $(\pi/2,\pi/2)$ is a saddle point.

(**OVER**)

2. Use the method of Lagrange multipliers to find the closest point to $P = (-3, 1)$ on the line $4x - 3y = -5$. (4 points)

SOLUTION:

Minimize $D = (x+3)^2 + (y-1)^2$, the function measuring distance from P to (x, y), subject to the restraint $g = 0$ where $g = 4x - 3y + 5$. We have $\nabla D = \langle 2(x+3), 2(y-2) \rangle$ and $\nabla g = \langle 4, -3 \rangle$. Using the method of Lagrange multipliers we get the system of equations:

$$(x+3) = 2\lambda$$
$$2(y-2) = -3\lambda$$

Solving for x and y in terms of λ we get $x = 2\lambda - 3$ and $y = -3/2\lambda + 2$. Plugging these values for x and y into $4x - 3y + 5 = 0$ we obtain that $\lambda = 4/5$ and so $x = -7/5$ and $y = -1/5$.

Math 241 - Quiz 2 - Tuesday, October 4

Your name here: **Version 3**

1. Let $f(x,y) = \cos x \sin y$.

   (a) Find $\nabla f(x,y)$. (1 point)
   SOLUTION:
   $\nabla f(x,y) = \langle -\sin x \sin y, \cos x \cos y \rangle$

   (b) Find all critical points of f and <u>use the second derivative test</u> to identify their types. (5 points)
   SOLUTION:
   The system $-\sin x \sin y = 0$ and $\cos x \cos y = 0$ is solved if either $\cos x = \sin y = 0$ or $\sin x = \cos y = 0$. Notice that we cannot have $\cos x = 0$ and $\sin x = 0$ since $\cos^2 x + \sin^2 y = 1$. We have:

$$\cos x = \sin y = 0 \Rightarrow \begin{cases} x = \frac{\pi}{2} + n\pi & n \text{ is any integer} \\ y = m\pi & m \text{ is any integer} \end{cases} \tag{1}$$

or

$$\sin x = \cos y = 0 \Rightarrow \begin{cases} x = n\pi & n \text{ is any integer} \\ y = \pi/2 + m\pi & m \text{ is any integer} \end{cases} \tag{2}$$

To use the second derivative test, we compute $f_{xx} = -\cos x \sin y$, $f_{yy} = -\cos x \sin y$, and $f_{xy} = -\sin x \cos y$. Call the Hessian D. We then have $D = f_{xx}f_{yy} - (f_{xy})^2 = (\cos x \sin y)^2 - (\sin x \cos y)^2$. In case (1) above $D = -(\sin x \cos y)^2 = -1 < 0$ (remember we have $\cos^2 x + \sin^2 x = 1$ so if $\cos x = 0$ then $\sin^2 x = 1$). So all the points in (1) are saddle points. In case (2) above $D = (\cos x \sin y)^2 = 1 > 0$, so we then analyze $f_{xx} = -\cos x \sin y$ for $x = n\pi$ and $y = \pi/2 + m\pi$. Notice that $\cos(n\pi)$ is 1 if n is even and -1 if n is odd and $\sin(\pi/2 + m\pi)$ is 1 if m is even and -1 if m is odd. This can be stated compactly by writing $\cos(n\pi) = (-1)^n$ and $\sin(\pi/2 + m\pi) = (-1)^m$. We have

$$\begin{aligned} f_{xx}(n\pi, \pi/2 + m\pi) &= -\cos(n\pi)\sin(\pi/2 + m\pi) \\ &= -(-1)^n(-1)^m \\ &= (-1)^{m+n+1} \end{aligned}$$

Hence $f_{xx} > 0$ when $m + n$ is odd and $f_{xx} < 0$ when $m + n$ is even. The table below summarizes these findings:

Point	min / max / saddle
$(n\pi, \pi/2 + m\pi)$ with $m + n$ odd	min
$(n\pi, \pi/2 + m\pi)$ with $m + n$ even	max
$(\pi/2 + n\pi, m\pi)$	saddle

For instance, if we restrict to the box $0 \leq x\pi$ and $-\pi/2 \leq y \leq \pi/2$ then we have 5 critical points: $(0, \pi/2)$ and $(\pi, -\pi/2)$ are both maximums, $(0, -\pi/2)$ and $(\pi, \pi/2)$ are both minimums, and $(\pi/2, 0)$ is a saddle point.

(**OVER**)

2. Use the method of Lagrange multipliers to find the closest point to $P = (-3, 1)$ on the line $4x - 3y = -10$. (4 points)

SOLUTION:

Minimize $D = (x+3)^2 + (y-1)^2$, the function measuring distance from P to (x, y), subject to the restraint $g = 0$ where $g = 4x - 3y + 10$. We have $\nabla D = \langle 2(x+3), 2(y-2) \rangle$ and $\nabla g = \langle 4, -3 \rangle$. Using the method of Lagrange multipliers we get the system of equations:

$$\begin{aligned} (x+3) &= 2\lambda \\ 2(y-2) &= -3\lambda \end{aligned}$$

Solving for x and y in terms of λ we get $x = 2\lambda - 3$ and $y = -3/2\lambda + 2$. Plugging these values for x and y into $4x - 3y + 10 = 0$ we obtain that $\lambda = 4/5$ and so $x = -11/5$ and $y = 7/5$.