SOLUTIONS TO WORKSHEET FOR TUESDAY, AUGUST 6, 2001

1. (a) Show that if $\mathbf{u}$ is any vector then $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

   SOLUTION:

   We have
   $$||\mathbf{u} \times \mathbf{u}|| = ||\mathbf{u}||^2 \sin\theta = ||\mathbf{u}||^2 \cdot 0 = 0$$
   since the angle between the $\mathbf{u}$ and itself is zero.

   (b) If $\mathbf{u}$ and $\mathbf{v}$ are any two nonzero vectors such that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, what can you say about the vectors $\mathbf{u}$ and $\mathbf{v}$?

   SOLUTION:

   $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\|\mathbf{u} \times \mathbf{v}\| = 0$ if and only if $\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = 0$ if and only if $\theta = 0$ or $\theta = \pi$. So $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

   (c) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors that are not parallel to each other. Show that the vector
   $$\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$$
   can never be zero.

   SOLUTION:

   Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $\mathbf{u} \times \mathbf{v} \neq 0$ by the solution to part *b*. Since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ (hence in particular not parallel to $\mathbf{u}$), we must have that $\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) \neq 0$ by part *b*.

2. Suppose that $\mathbf{v} \cdot \mathbf{w} = 0$. Find an expression for $\|\mathbf{v} \times \mathbf{w}\|$ (in terms of $\mathbf{v}$ and $\mathbf{w}$).

   SOLUTION:

   $\mathbf{v} \cdot \mathbf{w} = 0$ implies that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other. So
   $$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta = \|\mathbf{v}\|\|\mathbf{w}\| \sin 0 = \|\mathbf{v}\|\|\mathbf{w}\|.$$

3. (Volume of Prisms) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors. Consider the prism (parallelepiped) with vertex at the origin and with sides given by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The formula for the volume of a prism, like that of a cylinder, is
   $$\text{volume} = \text{base} \cdot \text{height}$$

   (a) Consider the face containing $\mathbf{u}$ and $\mathbf{v}$ as the "base", give the formula for the area of the base.

   SOLUTION:

   $\|\mathbf{u} \times \mathbf{w}\|$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{w}$, so in this case it is the area of the base.

(b) Find a formula for the height. This formula should be expressed in terms of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and the angle θ between $\mathbf{w}$ and the plane containing $\mathbf{u}$ and $\mathbf{v}$.

SOLUTION:

In terms of the angle θ between $\mathbf{w}$ and the plane containing $\mathbf{u}$ and $\mathbf{v}$, the height of the prism is $\|\mathbf{w}\| \sin\theta$. We can also find the height by taking the magnitude of the projection of $\mathbf{w}$ onto a normal vector to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. Note that $\mathbf{u} \times \mathbf{v}$ is such a normal vector, so we have

$$\begin{aligned} \text{height} =& \text{magnitude of the projection of } \mathbf{w} \text{ onto } \mathbf{u} \times \mathbf{v} \\ =& \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|} \end{aligned}$$

(c) Put your answers together to arrive at a formula for the volume of the prism.

SOLUTION:

Multiply the answers from a and b to obtain

$$\text{Volume} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

4. (A quadric surface in nonstandard form) Consider the surface described by the equation

$$4x^2 - 4xy + 4y^2 - 10x + 2y - 2z + 9 = 0.$$

(a) Introduce new variables $u = x + y$ and $v = x - y$. Solve for x and y in terms of u and v.

SOLUTION:

$$\begin{aligned} x =& \frac{u+v}{2} \\ y =& \frac{u-v}{2} \end{aligned}$$

(b) Substitute your answer above into the original equation to get a new equation in terms of u, v, and z.

SOLUTION:

After substituting and simplifying we obtain $u^2 - 4u + 3v^2 - 6v = 2z - 9$.

(c) Complete the square in both u and v to in order to arrive at a quadric equation in "standard form".

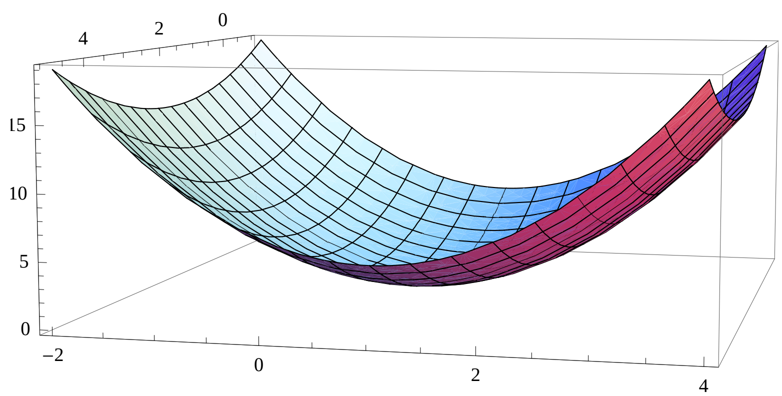
SOLUTION:

From completing the square we obtain $(u-2)^2 + 3(v-1)^2 = 2(z-1)$.

(d) What type of surface is this? If you don't remember the classification of quadric surfaces, try drawing the cross-sections with the standard coordinate planes.

SOLUTION:

When we set $z = c$ for $c > 0$ we obtain cross sections which are ellipses. If we fix u or v we get parabolas. The surface is an elliptic paraboloid (image on next page).



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