SOLUTIONS TO WORKSHEET FOR THURSDAY, AUGUST 8, 2001

1. Consider the function
$$f(x,y) = \frac{2xy}{x^2+y^2}.$$

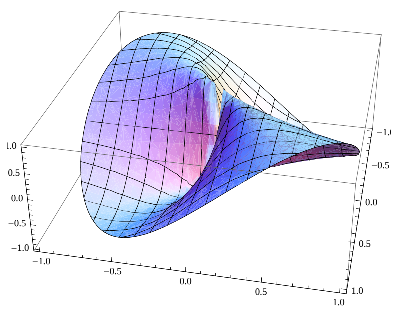
   (a) What does this function look like along a line $y = mx$?

   SOLUTION:

   Plugging in mx for y and simplifying we get $f(x, mx) = \frac{2m^2}{1+m^2}$. So f is constant along such lines.

   (b) Sketch the graph of $f(x,y)$.

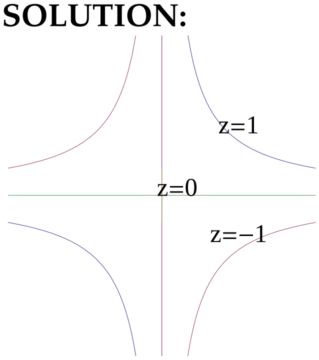
   SOLUTION:



2. Consider the function
$$f(x,y) = xy.$$

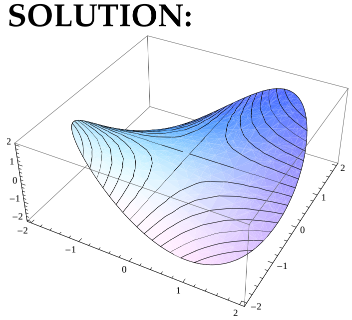
   (a) Sketch the level sets of f.

   SOLUTION:



   (b) Sketch the graph of $f(x,y)$. What is the name of this surface ?

   SOLUTION:



   This is a hyperbolic paraboloid.

3. Let $f(x,y) = 3x + 5y - 1$. This problems deals with

$$\lim_{(x,y)\to(1,1)} 3x + 5y - 1.$$

   (a) Let $\varepsilon = 1$. Find a $\delta > 0$ such that if $\|(x,y) - (1,1)\| < \delta$, then $|f(x,y) - 7| < \epsilon$.

   SOLUTION:

   Work backwards. We start with the inequality $|3x + 5y - 1 - 7| < 1$ or $-1 < 3(x - 1) + 5(y - 1) < 1$. Notice that if $-1/6 < (x-1) < 1/6$ and $-1/10 < (y-1) < 1/10$ then the inequality is satisfied. This gives a rectangle centered at $(1,1)$ so that if (x,y) is inside that rectangle, then $|f(x,y) - 7| < 1$. Now put a smaller circle centered at $(1,1)$ inside of that rectangle, say with radius $1/10$. If (x,y) is inside the circle, then it is also inside the rectangle. So $\delta = 1/10$ works.

   (b) Now find a $\delta > 0$ for arbitrary ε (your answer should be in terms of ε).

   SOLUTION:

   Follow exactly the steps of part *a* replacing 1 with ε. For the rectangle we get $-\varepsilon/6 < (x-1) < \varepsilon/6$ and $-\varepsilon/10 < (y-1) < \varepsilon/10$. So $\delta = \varepsilon/10$ works.

4. In class, we showed that

$$\lim_{(x,y)\to(1,0)} \frac{x}{y}$$

does not exist, by approaching the point $(1,0)$ along different lines. This can also be shown directly from the ε, δ definition. To do this, for each possible real number L, you must show that the limit cannot be L.

   (a) Let L be any real number. For the value $\varepsilon = 1$, show that no matter which $\delta > 0$ is chosen, there is always a point (x,y) such that $\|(x,y) - (1,0)\| < \delta$ but $|\frac{x}{y} - L| \geq 1$. This shows that the limit is not L.

   (**Hint:** Take *any* value for x in the interval $(1-\delta, 1+\delta)$. Show that there is a value for y that makes the above inequalities true.)

   SOLUTION:

   We want δ to be small, so we may as well assume that $\delta < 1/2$. This makes sure that $x \neq 0$. Now we find (x,y) so that $\|(x,y) - (1,0)\| < \delta$ and $|x/y - L| > 1$. Let's see if we can do this assuming that $x = 1$. Then we must find a y with $|y| < \delta$ so that $|1/y - L| > 1$. It is pretty clear that we just need to make y small. If we take $0 < y < 1/(|L| + 1)$ then $1/y > |L| + 1$ and $1/y - L > |L| + 1 - L > 1$ just like we wanted. So if we take $x = 1$ and $y =$ the minimum of δ and $1/(|L| + 1)$, then we are guaranteed that $\|(x,y) - (1,0)\| < \delta$ and $|\frac{x}{y} - L| \geq 1$.

   (b) More generally, show that for *any* $\varepsilon > 0$, no good δ can be found.

   SOLUTION:

   Follow the steps as in part *a* replacing 1 by ε. Take $x = 1$ and $0 < y < \min(1/(|L| + \varepsilon), \delta)$. Then $\|(x,y) - (1,0)\| < \delta$ and $|\frac{x}{y} - L| \geq \varepsilon$