SOLUTIONS TO WORKSHEET FOR TUESDAY, SEPTEMBER 13, 2001

1. Let $f(x,y) = y^2 \ln(x^3+1) + \sqrt{y}$. Find the partial derivatives

$$f_x, \qquad D_2 f, \qquad \frac{\partial^2 f}{\partial x^2}, \qquad D_1 D_2 f, \qquad f_{yy}, \qquad \frac{\partial^2 f}{\partial y \partial x}.$$

SOLUTION:

$$f_x = \tfrac{3x^2y^2}{x^3+1} \qquad D_2 f = 2y \log\left(x^3+1\right) + \tfrac{1}{2\sqrt{y}} \qquad \tfrac{\partial^2 f}{\partial x^2} = \tfrac{6xy^2}{x^3+1} - \tfrac{9x^4y^2}{\left(x^3+1\right)^2}$$
$$D_1 D_2 f = \tfrac{6x^2 y}{x^3+1} \qquad f_{yy} = 2\log\left(x^3+1\right) - \tfrac{1}{4y^{3/2}} \qquad \tfrac{\partial^2 f}{\partial y \partial x} = \tfrac{6x^2y}{x^3+1}$$

2. (Harmonic functions) The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called the **Laplace equation**. Any function $u(x,y,z)$ satisfying the Laplace equation is called a **harmonic function**.

(a) Let $u(x,y) = e^{ax}\cos(by)$. Find u_{xx} and u_{yy}. What must be true of a and b in order for u to be harmonic of two variables?

SOLUTION:

$u_{xx} = a^2 e^{ax}\cos(by)$ and $u_{yy} = -b^2 e^{ax}\cos(by)$. $u_{xx} + u_{yy} = 0$ if and only if $a^2 = b^2$ or $|a| = |b|$.

(b) Show that $u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a harmonic function of three variables.

SOLUTION:

$$u_{xx} = \tfrac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} \quad u_{yy} = \tfrac{-x^2+2y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} \quad u_{zz} = \tfrac{-x^2-y^2+2z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

From this we can see that $u_{xx} + u_{yy} + u_{zz} = 0$ and u is harmonic.

3. (Counterexample to Clairaut) Let

$$f(x) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Find $f_x(0,y)$.

(b) Find $f_y(x,0)$.

(c) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

SOLUTION:

First, $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$. Similarly we see that $f_y(0,0) = 0$. When $(x,y) \neq (0,0)$,

$$f_x = \frac{y\left(x^4+4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2}$$

and

$$f_y = \frac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2}$$

Substituting, we see that $f_x(0,y) = -y$ and $f_y(x,0) = x$. Since we computed that $f_x(0,0) = f_y(0,0) = 0$, these equations are also valid when $x = y = 0$. Now

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

and

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

So $f_{xy}(0,0) \neq f_{yx}(0,0)$.

4. The wind-chill index $W = f(T,v)$ is the perceived temperature when the actual temperature is T and the wind speed is v. Here is a table of values for W.

Wind speed (km/h)

Actual temperature (°C): T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

(a) Use the table to estimate $\frac{\partial f}{\partial T}$ and $\frac{\partial f}{\partial v}$ at $(T,v) = (-20,40)$.

SOLUTION:

$\frac{\partial f}{\partial T}\big|_{(-20,40)} \approx \frac{f(-15,40)-f(-20,40)}{-15+20} = 7/5$ and $\frac{\partial f}{\partial v}\big|_{(-20,40)} \approx \frac{f(-20,50)-f(-20,40)}{50-40} = -1/10$.

(b) Use your answer in (a) to write down the linear approximation to f at $(-20,40)$.

SOLUTION:

$$\begin{aligned} L &= f(-20,40) + \tfrac{\partial f}{\partial T}\big|_{(-20,40)}(T+20) + \tfrac{\partial f}{\partial v}\big|_{(-20,40)}(v-40) \\ &\approx -34 + 7/5(T+20) - 1/10(v-40) \end{aligned}$$

(c) Use your answer in (b) to approximate $f(-22,45)$. **SOLUTION:**

$$f(-22,45) \approx -34 + 7/5(-2) - 1/10(5) = -373/10 = -37.3$$