1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function having partial derivatives of all orders. The **Taylor series** of f centered around $\mathbf{c} = (a, b)$ is a power series in x and y of the form

$$T(f, \mathbf{c}) = f(\mathbf{c}) + \alpha_{1,0}(x-a) + \alpha_{0,1}(y-b)$$
$$+ \alpha_{2,0}(x-a)^2 + \alpha_{1,1}(x-a)(y-b) + \alpha_{0,2}(y-b)^2 + \text{higher order terms}$$

   (a) Assume that the Taylor series converges to f, so that

$$f(x,y) = T(f, \mathbf{c})(x,y)$$

   (at least in a disk around $\mathbf{c}$). Take partial derivatives of both sides with respect to x to find the coefficient $\alpha_{1,0}$. Use $\frac{\partial}{\partial y}$ to find $\alpha_{0,1}$.

   SOLUTION:
   $\dfrac{\partial T}{\partial x} = \alpha_{1,0} + 2\alpha_{2,0}(x-a) + \alpha_{1,1}(y-b) +$ higher order terms. Plugging in $x = a, y = b$ we see that all terms vanish except $\alpha_{1,0}$. So $\dfrac{\partial T}{\partial x}|_{(a,b)} = \alpha_{1,0}$. In the same way we find that $\dfrac{\partial T}{\partial y}|_{(a,b)} = \alpha_{0,1}$.

   (b) Use second order partial derivatives to find the coefficients $\alpha_{2,0}$, $\alpha_{1,1}$, and $\alpha_{0,2}$.

   SOLUTION:
   $\dfrac{\partial^2 T}{\partial x^2} = 2\alpha_{2,0} +$ higher order terms involving $(x-a)$ and $(y-b)$. Again, plugging in $x = a, y = b$, everything vanishes except $2\alpha_{2,0}$. So $\dfrac{\partial^2 T}{\partial x^2}|_{(a,b)} = 2\alpha_{2,0}$. In the same way we find that $\dfrac{\partial^2 T}{\partial x \partial y}|_{(a,b)} = \alpha_{1,1}$ and $\dfrac{\partial^2 T}{\partial y^2}|_{(a,b)} = 2\alpha_{0,2}$.

2. Consider $f(x,y) = 2\cos x - y^2 + e^{xy}$.

   (a) Show that $(0,0)$ is a critical point for f.

   SOLUTION:
   $\frac{\partial f}{\partial x}|_{(0,0)} = (-2\sin x + ye^{xy})|_{(0,0)} = 0$ and $\frac{\partial f}{\partial y} = (-2y + xe^{xy})|_{(0,0)} = 0$

   (b) Calculate each of f_{xx}, f_{xy}, f_{yy} at $(0,0)$ and use this to write out the 2^{nd}-order Taylor approximation for f at $(0,0)$.

   SOLUTION:
   $f_{xx} = -2\cos x + y^2 e^{xy}$, $f_{yy} = -2 + x^2 e^{xy}$, and $f_{xy} = e^{xy} + xye^{xy}$. So $f_{xx}(0,0) = -2 = f_{yy}(0,0)$ and $f_{xy}(0,0) = 1$. In the notation of problem 1 we have $\alpha_{1,0} = \alpha_{0,1} = 0$, $\alpha_{2,0} = \alpha_{0,2} = -1$, and $\alpha_{1,1} = 1$. Also $f(0,0) = 3$. So the second order Taylor approximation for f at $(0,0)$ is $g(x,y) = 3 - x^2 - y^2 + xy$.

   (c) To make sure the next two problems go smoothly, check your answer to (b) with the instructor.

   SOLUTION: Yes.

3. Let $g(x,y)$ be the approximation you obtained for $f(x,y)$ near $(0,0)$ in 1(b).

(a) It's not clear from the formula whether g, and hence f, has a min, max, or a saddle at $(0,0)$. Test along several lines until you are convinced you've determined which type it is.

SOLUTION:

Let's test a general line $y = mx$ which goes through $(0,0)$ as $x \to 0$. Then $g(x, mx) = 3 - x^2 - m^2x^2 + mx^2 = 3 - (1 - m + m^2)x^2$. The polynomial $1 - m + m^2$ is always positive (it opens upward and has its global minimum at $m = 1/2$ where $1 - m + m^2 > 0$). So $g(x, mx)$ is always a downward opening parabola. This suggests that $(0,0)$ is a relative maximum.

(b) Check that you're right in (a) using the 2^{nd}-derivative test. The next problem will help explain why this test works.

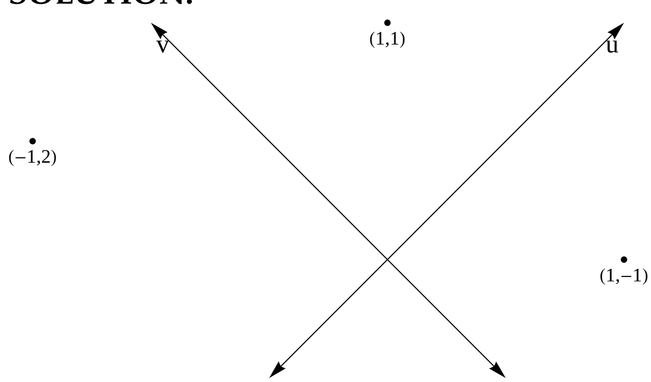
SOLUTION:

The Hessian $f_{xx}f_{yy} - (f_{xy})^2$ is $(-2)(-2) - 1^2 = 3 > 0$ at $(0,0)$ and $f_{xx}(0,0) = -2 < 0$. So f has a relative maximum at $(0,0)$ as suspected.

4. Consider alternate coordinates on $\mathbb{R}^2$ where (u, v) corresponds to $u(1,1) + v(-1,1)$.

(a) Sketch the u- and v-axes, and draw the points whose (u, v)-coordinates are: $(-1,2)$, $(1,1)$, $(1,-1)$.

SOLUTION:



(b) Give the general formula for the (x, y)-coordinates of a point in terms of u and v. (Like $x = r\cos\theta$ and $y = r\sin\theta$ in polar coordinates.)

SOLUTION:

Break the vectors into components. This gives $x = u - v$ and $y = u + v$.

(c) Use (b) to express g as a function of u and v, and expand and simplify the resulting expression.

SOLUTION:

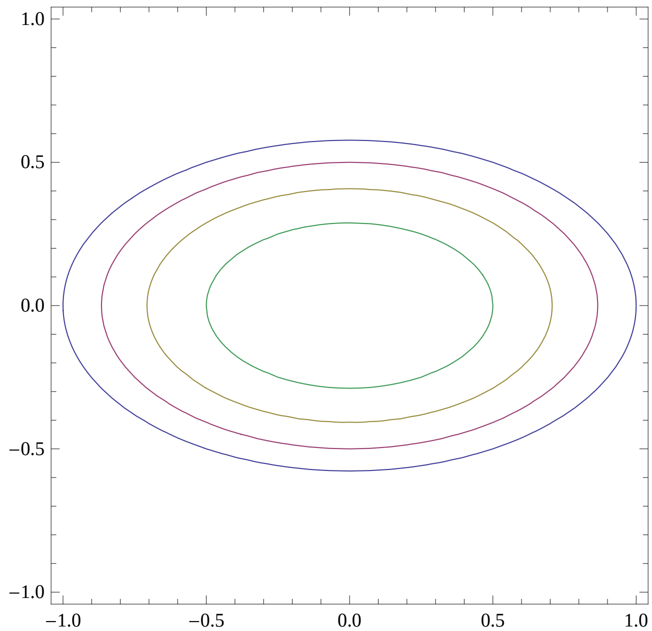
$3 - x^2 - y^2 + xy = 3 - (u - v)^2 - (u + v)^2 + (u - v)(u + v) = 3 - (u^2 - 2uv + v^2) - (u^2 + 2uv + v^2) + u^2 - v^2 = 3 - u^2 - 3v^2$.

(d) Explain why your answer in 3(c) confirms your answer in 2.

This is an elliptic paraboloid (in uv coordinates) opening downward with maximum at $(0,0,3)$, so it confirms that $(0,0)$ is a local maximum ($(0,0)$ goes to $(0,0)$ under the transformation, so this reasoning makes sense).

(e) Sketch a few level sets for g. What do the level sets of f look like near $(0,0)$?

SOLUTION: The level sets are sketched for $g = 2, 2.25, 2.5$, and 2.75. These are ellipses and they shrink as they get closer to $g(x,y) = 3$, which consists of the single solution $(x,y) = (0,0)$.



It turns out that there is always a similar change of coordinates so that the Taylor series of a function f which has a critical point at $(0,0)$ looks like $f(u,v) \approx f(0,0) + au^2 + bv^2$.