SOLUTIONS TO WORKSHEET FOR THURSDAY, SEPTEMBER 29, 2011

1. Which of the following equations/inequalities describe closed and bounded regions?

   (a) $2x^2 + y^2 + 4z^2 < 9$ in $\mathbb{R}^3$.

   SOLUTION:

   This is the closed and bounded region inside the ellipsoid with equation $2x^2 + y^2 + 4z^2 = 9$.

   (b) $x^3 + y^3 = 8$ in $\mathbb{R}^2$.

   SOLUTION:

   This is closed but not bounded. Imagine taking $x = 1000$. There is a solution to $y^3 = 8 - 1000^3$. You could do this for any real number, so this curve is not bounded.

   (c) $y \leq 3x + 1$ and $y \geq -2x + 1$ in $\mathbb{R}^2$.

   SOLUTION:

   This is a pie slice of R^2 translated up by one unit. So it is closed but not bounded.

   (d) $x^2 + y^2 = 1$ in $\mathbb{R}^2$.

   SOLUTION:

   This is a circle. It is closed and bounded.

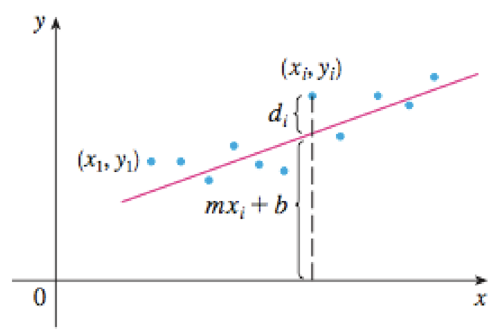
   (e) $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

   This is a cylinder with central axis the z-axis. It is closed but not bounded.

2. (Least squares approximations) Given a collection of data points

$$\{(x_1, y_1), \ldots, (x_n, y_n)\}$$

in the plane, the **least squares line** is the straight line that minimizes the sum of the squares of the vertical distances of the points from the line.



   (a) Find the formula for the sum of the squares of the vertical distances. Your formula should involve the points (x_i, y_i) and the constants m and b.

   SOLUTION:

   The square of the vertical distance from (x_i, y_i) to $mx + b$ is $(mx_i + b - y_i)^2$. So the sum of the squares of the vertical distances is

$$\sum_{i=1}^{n} (mx_i + b - y_i)^2 = (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \ldots + (mx_n + b - y_n)^2$$

   (b) For simplicity, assume $n = 3$. Thinking of the formula from part (a) as a function $f(m, b)$, find $\nabla(f)$.

SOLUTION:

The formula from part (a) for $n = 3$ is $(mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + (mx_3 + b - y_3)^2$. $\nabla(f) = \langle f_m, f_b \rangle$. We have

$$\begin{aligned} f_m =& 2(mx_1 + b - y_1)x_1 + 2(mx_2 + b - y_2)x_2 + 2(mx_3 + b - y_3)x_3 \\ =& 2m(x_1^2 + x_2^2 + x_3^2) + 2b(x_1 + x_2 + x_3) - 2(x_1y_1 + x_2y_2 + x_3y_3) \\ f_b =& 2(mx_1 + b - y_1) + 2(mx_2 + b - y_2) + 2(mx_3 + b - y_3) \\ =& 2m(x_1 + x_2 + x_3) + 6b - 2(y_1 + y_2 + y_3) \end{aligned}$$

(c) If the data points are $(1, 1)$, $(2, 2)$, and $(5, 3)$, where does this function attain a minimum? What is the resulting least squares line?

SOLUTION:

In this case we have $x_1 + x_2 + x_3 = 1 + 2 + 5 = 8$, $x_1^2 + x_2^2 + x_3^2 = 1^2 + 2^2 + 5^2 = 30$, $x_1y_1 + x_2y_2 + x_3y_3 = 1 + 4 + 15 = 20$, and $y_1 + y_2 + y_3 = 1 + 2 + 3 = 6$. From part (b) we have

$$\begin{aligned} f_m =& \ 2m(30) + 2b(8) - 2(20) &= 60m + 16b - 40 \\ f_b =& \ 2m(8) + 6b - 2(6) &= 16m + 6b - 12 \end{aligned}$$

Setting $f_m = f_b = 0$ we obtain a system of two linear equations in two unknowns:

$$\begin{aligned} 60m + 16b &= 40 \\ 16m + 6b &= 12 \end{aligned}$$

or

$$\begin{aligned} 15m + 4b &= 10 \\ 8m + 3b &= 6 \end{aligned}$$

Solving this we get $m = 6/13$ and $b = 10/13$. To check this is a minimum, the Hessian $f_{bb}f_{mm} - (f_{mb})^2 = 6 * 60 - 16^2 = 360 - 256 > 0$ and $f_{mm} = 60 > 0$ so $(m, b) = (6/13, 10/13)$ is indeed a minimum. So the resulting least squares line is $y = 6/13x + 10/13$.

3. Use Lagrange multipliers to find the point on the cone $z^2 = x^2 + y^2$ that is closest to the point $(4, 2, 0)$.

SOLUTION:

Minimize the square of the distance function $D = (x - 4)^2 + (y - 2)^2 + z^2$ from the point $(4, 2, 0)$ subject to the constraint $g = x^2 + y^2 - z^2 = 0$. We have $\nabla D = \langle 2(x - 4), 2(y - 2), 2z \rangle$ and $\nabla g = \langle 2x, 2y, -2z \rangle$. Using the method of Lagrange multipliers we have the system (divide out by 2 first):

$$
\begin{aligned}
(x-4) =& \lambda x \\
(y-2) =& \lambda y \\
z =& -\lambda z
\end{aligned}
$$

If $\lambda \neq -1$ then $z = 0$ from the last equation so the constraining equation $z^2 = x^2 + y^2$ implies that $x = y = 0$. If $\lambda = -1$ then the top two equations give $x = 2$ and $y = 1$. So the three possible points of minimum distance from $(4, 2, 0)$ are $(0, 0, 0)$, $(2, 1, \sqrt{5})$, and $(2, 1, -\sqrt{5})$. By calculation we see that the squares of the distances of each of these from $(4, 2, 0)$ are 20, 10, and 10 respectively. So the two points $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$ on the cone $z^2 = x^2 + y^2$ are of minimum distance from the point $(4, 2, 0)$.

4. Consider a rectangular box with sides of length a, b, and c. If the main diagonal of the box is of length L, what is the biggest possible volume of the box? Use Lagrange multipliers to maximize the volume function. (The diagonal length is the constraint.)

   SOLUTION:

   Set $x = a, y = b, z = c$. This simply supposes that the box is sitting in the octant $x \geq 0$, $y \geq 0$, and $z \geq 0$ with its edges along each axis. The volume function is then $V = xyz$ and the constraint is that $L^2 = x^2 + y^2 + z^2$. Using the method of Lagrange multipliers we get the system of equations:

$$
\begin{aligned}
yz =& 2\lambda x \\
xz =& 2\lambda y \\
xy =& 2\lambda z
\end{aligned}
$$

   Since we want to maximize volume we can assume that $x > 0, y > 0$, and $z > 0$. This rules out the possibility $\lambda = 0$ (since $\lambda = 0$ implies at least two of the variables x, y, and z are 0). Also this means we can multiply the first equation by x, the second by y, and the third by z to get a new system:

$$
\begin{aligned}
xyz =& 2\lambda x^2 \\
xyz =& 2\lambda y^2 \\
xyz =& 2\lambda z^2
\end{aligned}
$$

   This implies that $x^2 = y^2 = z^2$. Coupling this with the constraints $x > 0, y > 0, z > 0$ we see that this means $x = y = z$. Plugging this into the constraining equation $L^2 = x^2 + y^2 + z^2$ we get that $L^2 = 3x^2$ or $x = L/\sqrt{3}$. So $V = (L/\sqrt{3})^3 = L^3/(3\sqrt{3})$ is the biggest possible volume for the box.