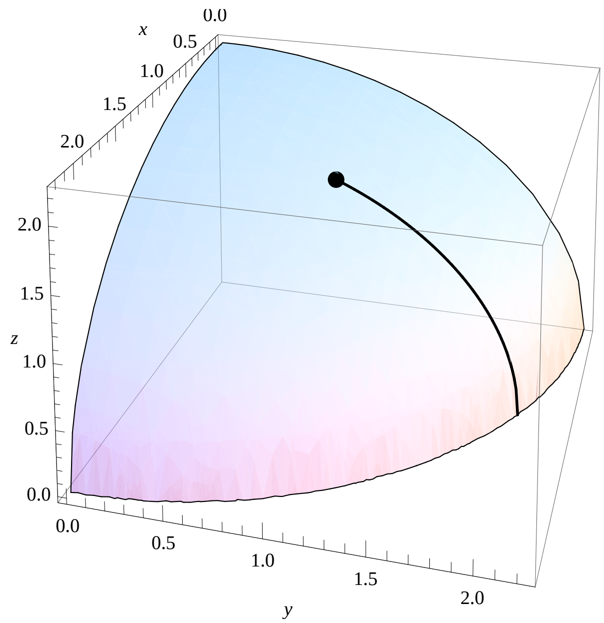
SOLUTIONS TO WORKSHEET FOR TUESDAY, OCTOBER 6, 2011

1. (a) Sketch the first-octant portion of the sphere $x^2 + y^2 + z^2 = 5$. Check that $P = (1, 1, \sqrt{3})$ is on this sphere and add this point to your picture.

SOLUTION:

$1^2 + 1^2 + (\sqrt{3})^2 = 5$ so this is on the sphere.



(b) Find a function $f(x, y)$ whose graph is the top-half of the sphere.

SOLUTION:

$f(x, y) = \sqrt{5 - x^2 - y^2}$

(c) Imagine an ant walking along the surface of the sphere. It walks *down* the sphere along the path C that passes through the point P in the direction parallel to the yz-plane. Draw this path in your picture.

SOLUTION:

See above.

(d) Use the function from (b) to find a parameterization $\mathbf{r}(t)$ of the ant's path along the portion of the sphere shown in your picture. Specify the domain for $\mathbf{r}$, i.e. the initial time when the ant is at P and the final time when it hits the xy-plane.

SOLUTION:

$x = 1$ along this path and $f(1, y) = \sqrt{4 - y^2}$, so setting $y = t$ we get the parametrization

$$\mathbf{r}(t) = (1, t, \sqrt{4 - t^2})$$

2. Consider the curve C in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = \left(e^t \cos t\right) \mathbf{i} + 2\mathbf{j} + \left(e^t \sin t\right) \mathbf{k}$$

(a) Calculate the length of the segment of C between $\mathbf{r}(0)$ and $\mathbf{r}(t_0)$. Check your answer with the instructor.

SOLUTION:

Length$=\int_0^{t_0} |r'(t)|dt = \int_0^{t_0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$. We have $\frac{dx}{dt} = e^t(\cos t - \sin t)$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = e^t(\sin t + \cos t)$, so

$$\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= e^{2t}((\cos t - \sin t)^2 + (\sin t + \cos t)^2) \\
&= e^{2t}(2\cos^2 t + 2\sin^2 t - 2\cos t \sin t + 2\cos t \sin t) \\
&= 2e^{2t}
\end{aligned}$$

So

$$\int_0^{t_0} |\mathbf{r}'(t)|dt = \int_0^{t_0} \sqrt{2e^{2t}}dt = \int_0^{t_0} e^t\sqrt{2}dt = \sqrt{2}(e^{t_0} - 1)$$

(b) Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is a function. We can get another parameterization of C by considering the composition

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

This is called a *reparameterization*. Find a choice of h so that

i. $\mathbf{f}(0) = \mathbf{r}(0)$

ii. The length of the segment of C between $\mathbf{f}(0)$ and $\mathbf{f}(s)$ is s. (This is called parameterizing by arc length.)

Check your answer with the instructor.

SOLUTION:

These two properties tell us that s needs to be $\int_0^t |\mathbf{r}'(u)|du$. From our computation in (a), $s = \sqrt{2}(e^t - 1)$. Since $\mathbf{r}$ is in terms of t, our function $h(s)$ is going to be the function that gives s in terms of t, i.e. $h(s) = t$. We get this by solving for t in the equation $s = \sqrt{2}(e^t - 1)$, so $h(s) = \ln(s/\sqrt{2} + 1)$.

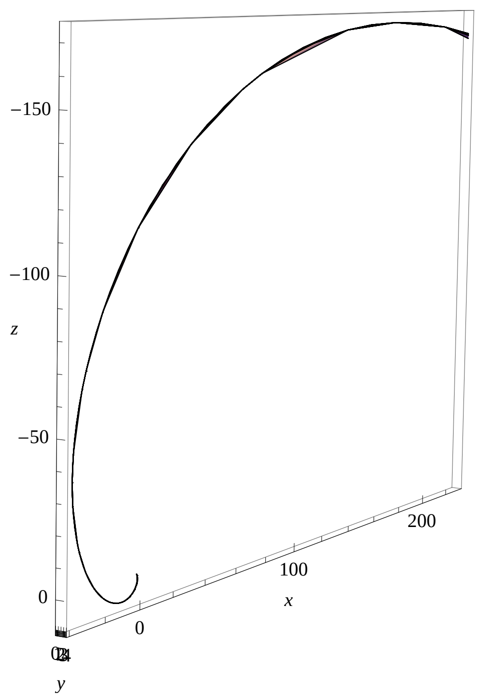
(c) Without calculating anything, what is $|\mathbf{f}'(s)|$?

SOLUTION:

Remember $s(t) = \int_0^t |\mathbf{r}'(u)|du$ so by the fundamental theorem of calculus, $s'(t) = |\mathbf{r}'(t)|$. Now by the chain rule $\mathbf{r}'(t) = \mathbf{f}'(s(t))s'(t)$. Taking magnitudes of both sides gives $|\mathbf{r}'(t)| = |\mathbf{f}'(s(t))| \cdot |s'(t)|$. By the first line $s'(t) = |\mathbf{r}'(t)|$. This gives that $|\mathbf{f}'(s(t))| = 1$. So $|\mathbf{f}'(s)| = 1$.

(d) Draw a sketch of C.

SOLUTION:



3. Consider the curve C given by the parameterization $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$ where $\mathbf{r}(t) = (\sin t, \cos t, \sin^2 t)$.

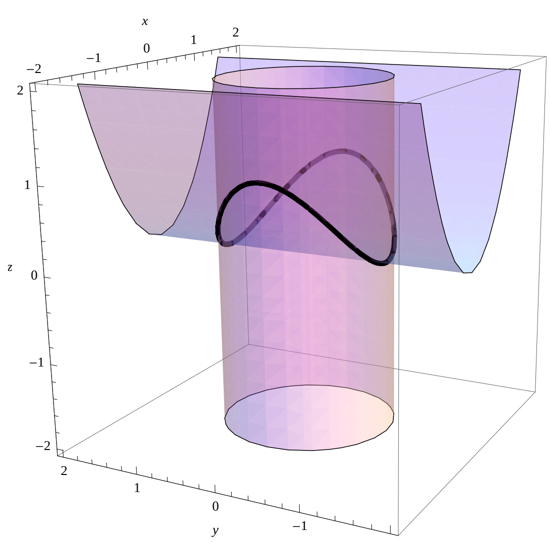
   (a) Show that C is in the intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

   SOLUTION:

   The z coordinate of $\mathbf{r}(t)$ is the square of the x-coordinate. Also the sum of the squares of the x and y coordinates of $\mathbf{r}(t)$ is $\sin^2 t + \cos^2 t = 1$ so $\mathbf{r}(t)$ is in the intersection of these two surfaces.

   (b) Use (a) to help you sketch the curve C.

   SOLUTION:



4. As in 2(b), consider a reparameterization

$$\mathbf{f}(s) = \mathbf{r}\big(h(s)\big)$$

of an arbitrary vector-valued function $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|\mathbf{f}'(s)|$ in terms of $\mathbf{r}'$ and h'.

SOLUTION:

$\mathbf{f}'(s) = \mathbf{r}'(h(s))h'(s)$ by the chain rule. Taking magnitudes of both sides we have $|\mathbf{f}'(s)| = |\mathbf{r}'(h(s))| \cdot |h'(s)|$.