SOLUTIONS TO WORKSHEET FOR TUESDAY, OCTOBER 11, 2011

1. Consider the function $f(x,y) = 8y$. Let C be the curve $x = y^2 - 1$ between the points $P = (-1,0)$ and $Q = (0,1)$.

   (a) Find a parametrization $\mathbf{r}(t)$ for C starting at P and ending at Q.

   SOLUTION:

   $\mathbf{r}(t) = \langle t^2 - 1, t\rangle, 0 \le t \le 1$.

   (b) Using the parametrization from (a), compute $\int_C f\,ds$.

   SOLUTION:

   $\int_C f\,ds = \int_0^1 (8t)\sqrt{1+4t^2}dt = \int_1^5 u^{1/2}du = 2/3(5^{3/2} - 1)$, where $u = 1 + 4t^2$.

   (c) Find a parametrization $\mathbf{q}(t)$ for C starting at Q and ending at P, and use this to calculate $\int_C f\,ds$. Did you get the same answer as in (b)?

   SOLUTION:

   $\mathbf{q}(t) = \langle (1-t)^2 - 1, 1-t\rangle, 0 \le t \le 1$. For this parametrization,

$$\int_C f\,ds = \int_0^1 8(1-t)\sqrt{1+4(1-t)^2}dt = \int_1^5 u^{1/2}du = 2/3(5^{3/2} - 1)$$

   where $u = 1 + 4(1-t)^2$. This is the same answer as in (b).

   (d) Using the two parametrizations from (a) and (c), calculate $\int_C f\,dy$. Do you get the same answer in both cases?

   SOLUTION:

   For the parametrization from (a) we have

$$\int_C f\,dy = \int_0^1 8tdt = 4$$

   while for the parametrization from (c) we have

$$\int_C f\,dy = \int_0^1 8(1-t)(-dt) = -4.$$

   So the answers differ by a sign change.

   (e) Write $\mathbf{F} = \nabla(f)$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ and $\int_C \mathbf{F}\cdot d\mathbf{q}$. Do you get the same answer in both cases?

   SOLUTION:

   $\mathbf{F} = \nabla(F) = \langle 0, 8\rangle$. We have

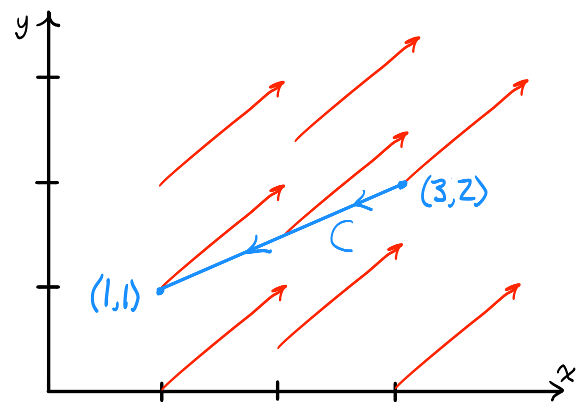
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 0,8\rangle \cdot \langle 2t, 1\rangle\,dt = 8$$

   and

$$\int_C \mathbf{F} \cdot d\mathbf{q} = \int_0^1 \langle 0, 8 \rangle \cdot \langle 2(1-t), -1 \rangle = -8.$$

So these answers also differ by a sign.

2. Consider the curve C and vector field $\mathbf{F}$ shown to the right ($\mathbf{F}(x,y) = \langle 1, 1 \rangle$).



   (a) Without parameterizing C, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Hint: Use the Fundamental Theorem for Line Integrals.)

   SOLUTION:

   Set $f = x + y$. Then $\mathbf{F} = \nabla(f)$. Using the Fundamental Theorem for Line Integrals we have $\int_C \nabla(f) \cdot d\mathbf{r} = f(1,1) - f(3,2) = 2 - 5 = -3$.

   (b) Find a parameterization of C and use it to check your answer in (a) by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$ explicitly.

   SOLUTION:

   Parametrize C by $\mathbf{r} = \langle 3 - 2t, 2 - t \rangle$, $0 \le t \le 1$. So $r'(t) = \langle -2, -1 \rangle$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 1, 1 \rangle \cdot \langle -2, -1 \rangle \, dt = \int_0^1 -3dt = -3$.

3. Consider the vector field $\mathbf{F} = (-y, x)$.

   (a) Let C_1 be the straight line segment from $(1,0)$ to $(-1,0)$. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

   SOLUTION:

   Parametrize C_1 by $\mathbf{r}(t) = (0, t)$, $-1 \le t \le 1$. We have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^1 (0, t) \cdot (1, 0) dt = 0$$

   (b) Let C_2 be the upper semicircle (with counter-clockwise orientation). Compute $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

   SOLUTION:

   Parametrize C_2 by $\mathbf{q}(t) = (\cos t, \sin t)$, $0 \le t \le \pi$. We have

$$\int_{C_2} \mathbf{F} \cdot \mathbf{q} = \int_0^\pi (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^\pi 1 dt = \pi.$$

   NOTE: This indicates that the vector field is not conservative on any open set containing the two paths C_1 and C_2.

4. Suppose the curve C is contained within a level set of the function f.

   (a) From the point of view of the Fundamental Theoerm, why is $\int_C \nabla(f) \cdot d\mathbf{r} = 0$?

   SOLUTION:

   We will assume $f(x,y)$ is a function of two variables for simplicity. Let C have initial point (x_0, y_0) and final point (x_1, y_1). Since C is contained in a level curve of f we have $f(x_0, y_0) = f(x_1, y_1)$. By the Fundamental Theorem of Line Integrals,

$$\int_C \nabla(f) \cdot d\mathbf{r} = f(x_1, y_1) - f(x_0, y_0) = 0.$$

   (b) Give a geometric explanation, not using the Fundamental Theorem, for why this line integral should be zero.

   Again assume f is a function of 2 variables and that C has initial point (x_0, y_0) and final point (x_1, y_1). Remember $\int_C \mathbf{F} \cdot d\mathbf{r}$ can be interpreted as work done by the force field $\mathbf{F}$ on a particle moving from (x_0, y_0) to (x_1, y_1). But the force field $\nabla(f)$ is always perpendicular to C since C lies in a level curve of f (remember $\nabla(f)(a,b)$ is perpendicular to the level curve $f(x,y) = f(a,b)$ for every point (a,b) in the domain of f and $\nabla(f)$). So the force field does no work as the particle moves along C.