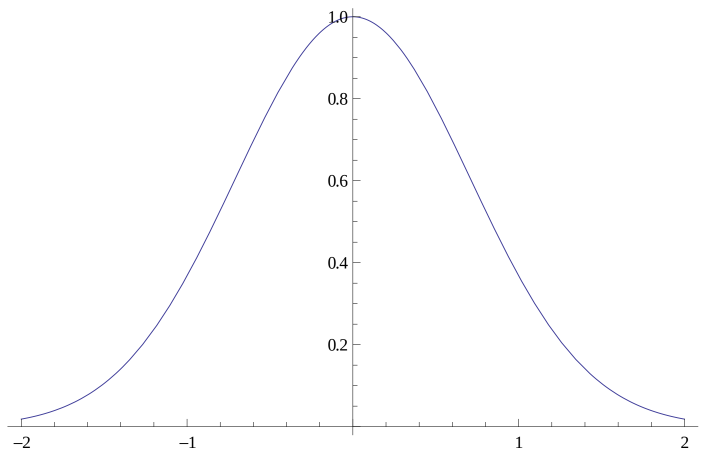
SOLUTIONS TO WORKSHEET FOR TUESDAY, OCTOBER 25, 2011

1. The function $P(x) = e^{-x^2}$ is fundamental in probability.

   (a) Sketch the graph of $P(x)$. Explain why it is called a "bell curve."
   SOLUTION:



   (b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ using the following brilliant strategy of Gauss.

      i. Instead of computing I, compute $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

      ii. Rewrite I^2 as an integral of the form $\iint_R f(x,y)\,dA$ where R is the entire Cartesian plane.
      SOLUTION:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx$$

      iii. Convert that integral to polar coordinates.
      SOLUTION:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta$$

      iv. Evaluate to find I^2. Deduce the value of I.
      SOLUTION:

$$\int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-r^2}\,dr = 2\pi\lim_{t\to\infty}\int_0^{t} re^{-r^2}dr = 2\pi\lim_{t\to\infty}\left[-1/2e^{-r^2}\right]_0^t$$

$$= \pi\lim_{t\to\infty}(-e^{-t^2}+1) = \pi$$

      So $I = \sqrt{\pi}$.

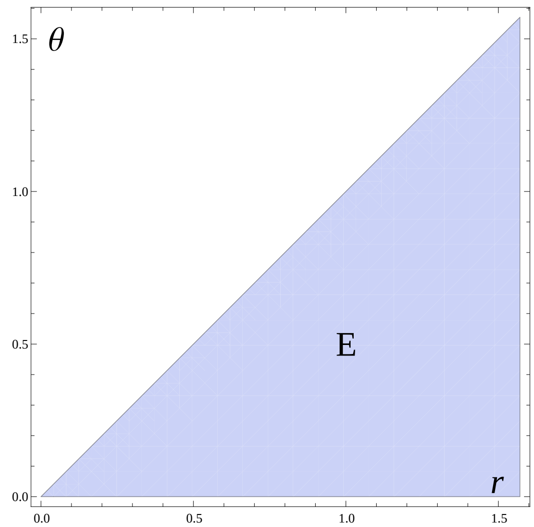
Amazingly, it can be mathematically proven that there is NO elementary function $Q(x)$ (that is, function built up from sines, cosines, exponentials, and roots using "usual" operations) for which $Q'(x) = P(x)$.

2. Let E be the polar triangle

$$E = \{(r, \theta) \mid 0 \leq r \leq \pi/2, 0 \leq \theta \leq r\}.$$

(a) Sketch E and compute its area.

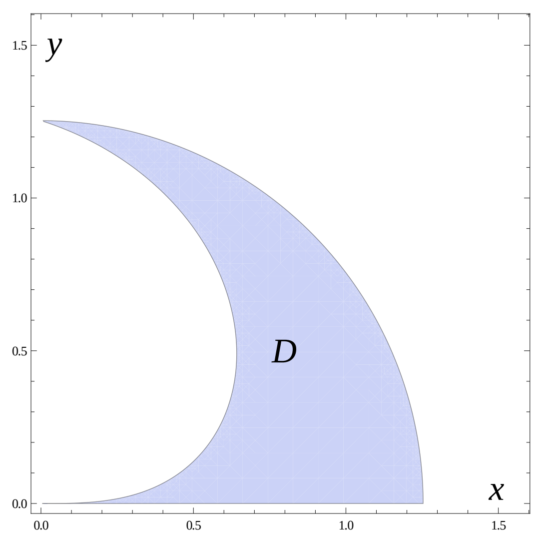
SOLUTION:



The area of this triangle in the r-θ plane is $1/2(\pi/2)(\pi/2) = \pi^2/8$.

(b) Let D be the region in the cartesian plane corresponding to E. Sketch D and find its area.

SOLUTION:



Area of D=

$$\int\int_D dxdy = \int\int_E r\,dr\,d\theta = \int_0^{\pi/2}\int_\theta^{\pi/2} r\,dr\,d\theta = \int_0^{\pi/2} \frac{\pi^2}{8} - \frac{\theta^2}{2} d\theta = \frac{\pi^3}{16} - \frac{(\pi/2)^3}{6}$$

3. We have discussed the fact that the area of a disc of radius r is πr^2 and that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

   (a) Use a quadruple integral to find the volume of the hypersphere

$$x^2 + y^2 + z^2 + w^2 = r^2$$

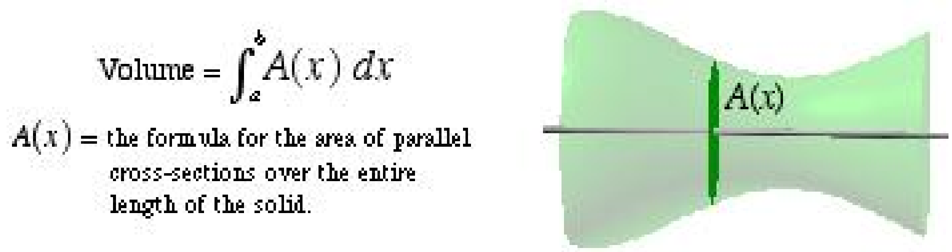
of radius r in $\mathbb{R}^4$. You may wish to use either of the following integration formulas:

$$\int \cos^4\theta \, d\theta = \frac{1}{16}\left[4\cos^3\theta\sin\theta + 6\theta + 3\sin 2\theta\right],$$

$$\text{or} \quad \int \sin^4\theta \, d\theta = \frac{1}{16}\left[-4\sin^3\theta\cos\theta + 6\theta - 3\sin 2\theta\right].$$

SOLUTION:

Recall from earlier in the calculus sequence that if we are given a function $A(x)$ which gives the area of a cross section of a 3 dimensional region D when x is fixed then we can compute the volume of D by the integral $\int_a^b A(x)\,dx$, where a and b are the least and greatest x coordinates appearing in the region D.



We can do the same to compute the volume of higher dimensional objects. When $w = k$ is fixed and $-r \le w \le r$ then x,y and z satisfy $x^2 + y^2 + z^2 = r^2 - k^2$, the equation of a 3 dimensional sphere of radius $\sqrt{r^2 - k^2}$. The volume of this sphere is given by $4/3\pi(\sqrt{r^2 - k^2})^3$. Hence the volume of the hypersphere in 4 space is given by

$$\int_{-r}^{r} 4/3\pi(\sqrt{r^2 - w^2})^3\,dw = 8\pi/3\int_0^r (\sqrt{r^2 - w^2})^3\,dw$$

We can use the trig substitution $w = r\sin\theta$ here and get:

$$8\pi/3\int_0^r (\sqrt{r^2 - w^2})^3\,dw = 8\pi/3\int_0^{\pi/2} (\sqrt{r^2 - r^2\sin^2\theta})^3 r\cos\theta\,d\theta = 8\pi/3r^4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

With the help of the formula above, we see that only the lonely θ term will contribute to the final integral, so $\int_0^{\pi/2}\cos^4\theta\,d\theta = 3\pi/16$. So our final formula is

$$8\pi/3r^4\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{\pi^2}{2}r^4$$

(b) Use an iterated integral to find the volume of the hypersphere of radius r in $\mathbb{R}^n$ to be

$$V_n = \frac{2^{(n+1)/2}}{3 \cdot 5 \cdot \cdots \cdot n} \pi^{(n-1)/2} r^n, \qquad n \text{ odd}$$
$$V_n = \frac{2^{n/2}}{2 \cdot 4 \cdot \cdots \cdot n} \pi^{n/2} r^n, \qquad n \text{ even.}$$

You may wish to use the reduction formula

$$\int \cos^n \theta \, d\theta = \frac{1}{n} \cos^{n-1} \theta \sin \theta + \frac{n-1}{n} \int \cos^{n-2} \theta \, d\theta$$
$$\text{or} \quad \int \sin^n \theta \, d\theta = \frac{-1}{n} \sin^{n-1} \theta \cos \theta + \frac{n-1}{n} \int \sin^{n-2} \theta \, d\theta.$$

SOLUTION:

These formulas may be established by induction using the slice method above. We have already done several cases that can serve as the base for the induction. In the induction step we need to treat the cases where n is even and odd separately.

CASE 1: If n is even then $n-1$ is odd and we have (using the trig substitution $x = r \sin \theta$)

$$V_n(r) = \int_{-r}^{r} V_{n-1}(x) \, dx = 2 \int_0^r \frac{2^{n/2}}{3 \cdot 5 \cdot \cdots \cdot (n-1)} \pi^{(n-2)/2} (\sqrt{r^2 - x^2})^{n-1} \, dx$$

$$= \frac{2^{(n+2)/2}}{3 \cdot 5 \cdot \cdots \cdot (n-1)} \pi^{(n-2)/2} r^n \int_0^{\pi/2} \cos^n \theta \, d\theta$$

By iteratively using the reduction formula given above we derive that if n is even then

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{3 \cdot 5 \cdot 7 \cdots (n-1)}{4 \cdot 6 \cdot 8 \cdots n} \frac{\pi}{4}$$

So

$$\frac{2^{(n+2)/2}}{3 \cdot 5 \cdot \cdots \cdot (n-1)} \pi^{(n-2)/2} r^n \int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{2^{(n+2)/2}}{3 \cdot 5 \cdot \cdots \cdot (n-1)} \pi^{(n-2)/2} r^n \cdot \frac{3 \cdot 5 \cdot 7 \cdots (n-1)}{4 \cdot 6 \cdot 8 \cdots n} \frac{\pi}{4}$$

$$= \frac{2^{n/2}}{2 \cdot 4 \cdot 6 \cdots n} \pi^{n/2} r^n$$

so the induction step is complete for n even.

CASE 2: If n is odd then $n-1$ is even and we have

$$V_n(r) = \int_{-r}^{r} V_{n-1}(x) \, dx = 2 \int_0^r \frac{2^{(n-1)/2}}{2 \cdot 4 \cdot \cdots \cdot (n-1)} \pi^{(n-1)/2} (\sqrt{r^2 - x^2})^{n-1} \, dx$$

$$= \frac{2^{(n+1)/2}}{2 \cdot 4 \cdot \cdots \cdot (n-1)} \pi^{(n-1)/2} r^n \int_0^{\pi/2} \cos^n \theta \, d\theta$$

Again iteratively using the reduction formula given above we derive that if n is odd then

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}$$

So

$$\frac{2^{(n+1)/2}}{2 \cdot 4 \cdot \cdots \cdot (n-1)} \pi^{(n-1)/2} r^n \int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{2^{(n+1)/2}}{2 \cdot 4 \cdot \cdots \cdot (n-1)} \pi^{(n-1)/2} r^n \cdot \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}$$

$$= \frac{2^{(n+1)/2}}{3 \cdot 5 \cdot 7 \cdots n} \pi^{(n-1)/2} r^n$$

so the induction step is complete for n odd and we are done.