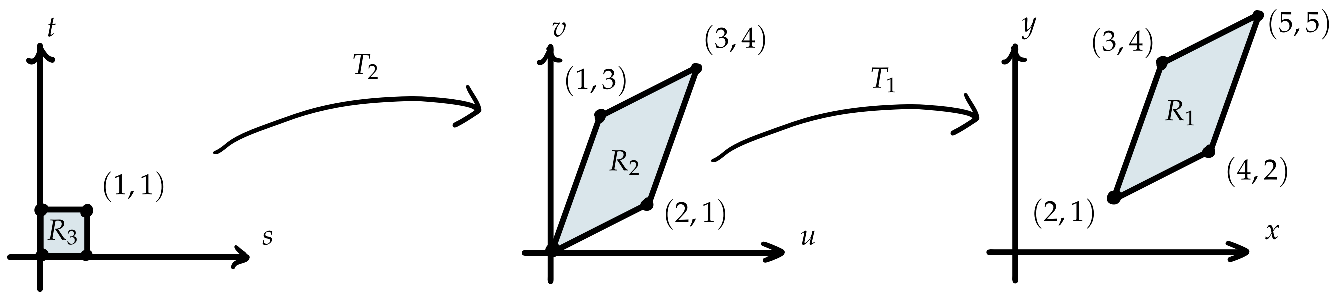
SOLUTIONS FOR THURSDAY, NOVEMBER 3

1. Consider the region R_1 in $\mathbb{R}^2$ shown below at right. In this problem, you will do a series of changes of coordinates to evaluate:

$$\iint_{R_1} x - 2y\, dA$$



(a) A simple type of transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(s,t) = (s+a, t+b)$ for a fixed a and b. Find a translation T_1 such that $T_1(R_2) = R_1$.

SOLUTION:

$$T_1(u,v) = (u+2, v+1)$$

(b) If T is a translation, what is its Jacobian matrix? How does it distort area?

SOLUTION:

If $T(u,v) = (u+a, v+b)$ where a and b are constants, then the Jacobian is

$$det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

So T does not distort areas.

(c) Rewrite the original integral in terms of an integral over R_2.

SOLUTION:

$$\iint_{R_1} x - 2y\, dA = \iint_{R_2} (u+2) - 2(v+1) J(T_1)\, dA = \iint_{R_2} u - 2v\, dA$$

(d) Find a linear transformation $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square R_3 to R_2. Check your answer with the instructor.

SOLUTION:

$$T_2(s,t) = (2s+t, s+3t)$$

(e) Compute $\iint_{R_1} x - 2y\, dA$ by relating it to an integral over R_3 and evaluating that. Check your answer with the instructor.

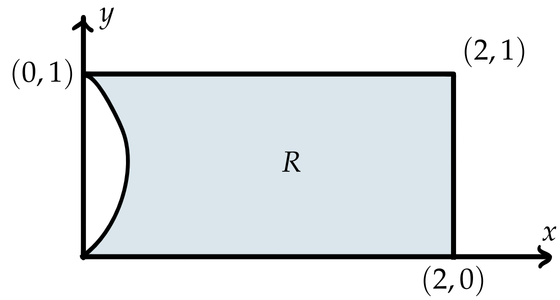
SOLUTION:

The Jacobian of T_2 is 5. So

$$\iint_{R_1} x - 2y\, dA = \iint_{R_2} u - 2v\, dA = \int_0^1 \int_0^1 (2s + t) - 2(s + 3t) J(T_2)\, ds\, dt$$

$$= \int_0^1 \int_0^1 -25t\, ds\, dt = -25/2$$

2. Consider the region R shown below. Here the curved left side is given by $x = y - y^2$. In this problem, you will find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S = [0,1] \times [0,1]$ to R.



(a) As a warm up, find a transformation that takes S to the rectangle $[0,2] \times [0,1]$ which contains R.

SOLUTION:

$$L(u,v) = (2u, v)$$

(b) Returning to the problem of finding T taking S to R, come up with formulas for $T(u,0)$, $T(u,1)$, $T(0,v)$, and $T(1,v)$. Hint: For three of these, use your answer in part (a).

SOLUTION:

$$T(u,0) = (2u,0) \quad T(u,1) = (2u,1)$$
$$T(1,v) = (2,v) \quad T(0,v) = (v - v^2, v)$$

(c) Now extend your answer in (b) to the needed transformation T. Hint: Try "filling in" between $T(0,v)$ and $T(1,v)$ with a straight line.

SOLUTION:

$$T(u,v) = (2u + v(1-v)(1-u), v)$$

(d) Compute the area of R in two ways, once using T to change coordinates and once directly.

SOLUTION:

To change coordinates we compute the Jacobian

$$J(T) = det \begin{pmatrix} 2 - v(1-v) & (1-2v)(1-u) \\ 0 & 1 \end{pmatrix} = 2 - v(1-v)$$

So we have the area of R given by

$$\iint_R dx\, dy = \int_0^1 \int_0^1 2 - v(1-v)\, du\, dv = 11/6$$

Computing directly we have the area of R given by

$$\int_0^1 2 - (y - y^2)\, dy = 11/6$$

3. In order to do a change of coordinates in three variables, you need to compute a determinant of a 3×3 Jacobian matrix. In this problem, you will practice computing such determinants. Consider the 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) One method of computing 3×3 determinants is by considering the diagonals. The determinant of A can be computed as follows: add up the products along the three $\searrow$ diagonals and subtract off the products along the three $\swarrow$ diagonals. Find $\det B$.

SOLUTION:

Using this method we have $\det A = aei + bfg + cdh - afh - bdi - ceg$. Applying this rule to B we find that

$$\det B = -2 + 0 + 0 - 0 - 6 - 2 = -10$$

(b) The "cofactor" method of computing determinants is as follows: pick a row of the matrix. For each entry on that row, multiply that entry by the determinant of the 2×2 matrix obtained by removing that row and column from the 3×3 matrix. The determinant is then the *alternating* sum of these products (alternating means that every other term has a negative sign). If you use the first or third row, the signs are $+ \; - \; +$, and if you use the second row, the signs are $- \; + \; -$.

For example, using the first row,

$$\det A = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

Compute $\det B$ by the method of cofactors once for each row. As you can see, it is usually a good idea to pick a row with the most 0's.

SOLUTION:

Using row 1:

$$\det B = 1 \cdot (-2) - 2 \cdot (3) + (-1) \cdot 2 = -10$$

Using row 2:

$$\det B = -3 \cdot (2) + (-2) \cdot (2) - 0 = -10$$

Using row 3:

$$\det B = 1 \cdot (-2) - 0 + 1 \cdot (-8) = -10$$