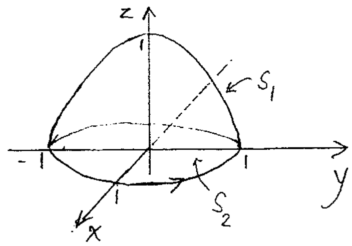
Worksheet 15 Solutions : 11/29/11

1. (a)



1.(b) $S_1: \vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2)$,

$0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

Then $\vec{n} = \vec{r}_r \times \vec{r}_\theta = (2r^2\cos\theta, 2r^2\sin\theta, r)$.

So we get
$$\iint_{S_1} \vec{F}\cdot\vec{n}\, dS = \int_0^{2\pi}\int_0^1 (0,0,1-r^2)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta = \frac{\pi}{2}.$$

1. (c) The downward pointing normal vector is $\vec{n} = -\vec{k}$. Since $z=0$ on S_2, we get
$$\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iint_{S_2} (-z)\, dS = 0.$$

1. (d) By 1.(b)-(c), the flux of F through $\partial D = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \frac{\pi}{2} + 0 = \frac{\pi}{2}$.

1. (e) $\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial z}{\partial z} = 1$. By Divergence Theorem,

volume of $D = \iiint_D 1\, dV = \iiint_D \operatorname{div}(\vec{F})\,dV = \iint_{\partial D} \vec{F}\cdot d\vec{S} = \frac{\pi}{2}$ (by 1.(d)).

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2. (a) $$\operatorname{curl}(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & z \end{vmatrix} = (0,0,2)$$

2.(b) $$\iint_{S_1} (\operatorname{curl}(\vec{F}))\cdot\vec{n}\, dA = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = 2\pi.$$

2.(c) We have $C: \vec{r}(t) = (\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$.

$$\iint_{S_1} \operatorname{curl}(\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$
$$= \int_0^{2\pi} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t + 0)\,dt$$
$$= 2\pi.$$

3. (a) $V = \iiint_D 1\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r \, dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\,d\theta$$

$$= \Big[\theta\Big]_{\theta=0}^{\theta=2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{1} = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}.$$

3. (b) $\iint_{S_2} \text{curl}(\vec{F})\cdot\vec{n}\, dA = \int_0^{2\pi}\int_0^1 (0,0,2)\cdot(0,0,-1)\, r\, dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (-2r)\, dr\,d\theta$$

$$= -2\pi.$$

Using Stokes' Theorem,

$$\iint_{S_2} \text{curl}(\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_{2\pi}^{0} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\, dt$$

$$= \int_{2\pi}^{0} 1\, dt = -2\pi$$

So we get $\iint_{\partial D} \text{curl}(\vec{F})\cdot\vec{n}\, dA = 2\pi + (-2\pi) = 0.$

2.(c) means that if we have another oriented surface with the same boundary curve C, then we will get the same value for the surface integral.

3. (c) We have $\text{curl}(\vec{F}) = (0,0,2)$, so $\text{div}(\text{curl}\,\vec{F}) = 0.$

For arbitrary $\vec{F} = (F_1, F_2, F_3)$,

$$\text{curl}\,\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

Then $\text{div}(\text{curl}\,\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$= 0.$ by Clairaut's Theorem.