Tuesday, October 25 $**$ *Multiple integrals & Polar coordinates*

1. The function $P(x) = e^{-x^2}$ is fundamental in probability.

   (a) Sketch the graph of $P(x)$. Explain why it is called a "bell curve."

   (b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ using the following brilliant strategy of Gauss.

      i. Instead of computing I, compute $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

      ii. Rewrite I^2 as an integral of the form $\iint_R f(x,y)\,dA$ where R is the entire Cartesian plane.

      iii. Convert that integral to polar coordinates.

      iv. Evaluate to find I^2. Deduce the value of I.

   Amazingly, it can be mathematically proven that there is NO elementary function $Q(x)$ (that is, function built up from sines, cosines, exponentials, and roots using "usual" operations) for which $Q'(x) = P(x)$.

2. Let E be the polar triangle

$$E = \{(r,\theta) \mid 0 \le r \le \pi/2, 0 \le \theta \le r\}.$$

   (a) Sketch E and compute its area.

   (b) Let D be the region in the cartesian plane corresponding to E. Sketch D and find its area.

3. We have discussed the fact that the area of a disc of radius r is πr^2 and that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

   (a) Use a quadruple integral to find the volume of the hypersphere

$$x^2 + y^2 + z^2 + w^2 = r^2$$

   of radius r in $\mathbb{R}^4$. You may wish to use either of the following integration formulas:

$$\int \cos^4\theta\,d\theta = \frac{1}{16}\left[4\cos^3\theta\sin\theta + 6\theta + 3\sin 2\theta\right],$$

$$\text{or} \quad \int \sin^4\theta\,d\theta = \frac{1}{16}\left[-4\sin^3\theta\cos\theta + 6\theta - 3\sin 2\theta\right].$$

   (b) Use an iterated integral to find the volume of the hypersphere of radius r in $\mathbb{R}^n$ to be

$$V_n = \frac{2^{(n+1)/2}}{3\cdot 5\cdot\cdots\cdot n}\pi^{(n-1)/2}r^n, \qquad n \text{ odd}$$

$$V_n = \frac{2^{n/2}}{2\cdot 4\cdot\cdots\cdot n}\pi^{n/2}r^n, \qquad n \text{ even.}$$

   You may wish to use the reduction formula

$$\int \cos^n\theta\,d\theta = \frac{1}{n}\cos^{n-1}\theta\sin\theta + \frac{n-1}{n}\int \cos^{n-2}\theta\,d\theta$$

$$\text{or} \quad \int \sin^n\theta\,d\theta = \frac{-1}{n}\sin^{n-1}\theta\cos\theta + \frac{n-1}{n}\int \sin^{n-2}\theta\,d\theta.$$