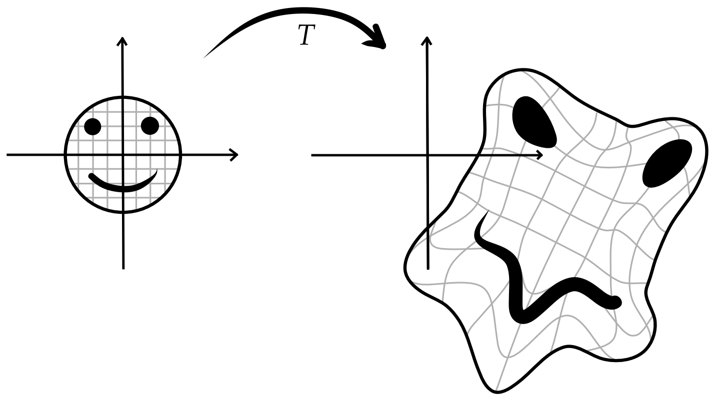
Thursday, October 27 $**$ *Transformations of* $\mathbb{R}^2$.

Purpose: In class, we've seen several different coordinate systems on $\mathbb{R}^2$ and $\mathbb{R}^3$ beyond the usual rectangular ones: polar, cylindrical, and spherical. The lectures on Friday and Monday will cover the crucial technique of simplifying hard integrals using a change of coordinates (Section 15.9). The point of this worksheet is to familiarize you with some basic concepts and examples for this process.

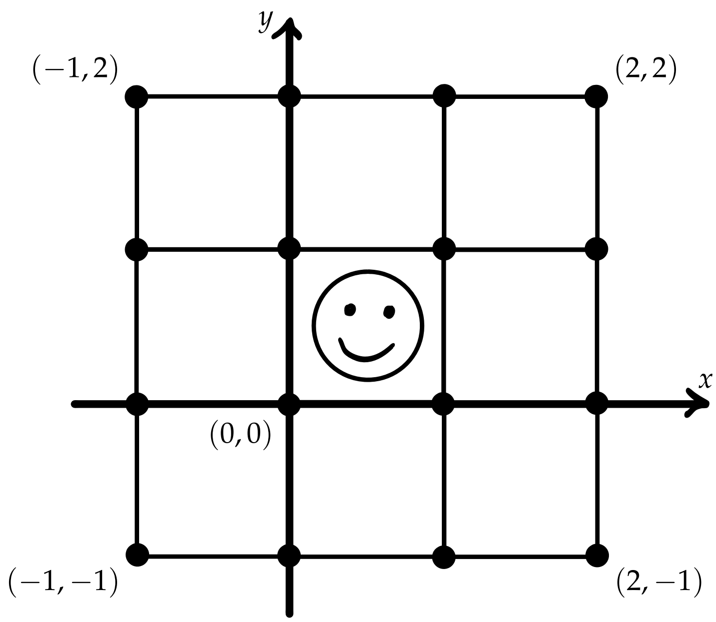
Starting point: Here we consider a variety of transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. Previously, we have used such functions to describe vector fields on the plane, but we can also use them to describe ways of distorting the plane:



1. Consider the transformation $T(x, y) = (x - 2y, x + 2y)$.

   (a) Compute the image under T of each vertex in the below grid and make a careful plot of them, which should be fairly large as you will add to it later.

   To speed this up, divide the task up among all members of the group.



(b) For each pair A and B of vertices of the grid joined by a line, add the line segment joining $T(A)$ to $T(B)$ to your plot. This gives a rough picture of what T is doing.

Check your answer with the instructor.

(c) What is the image of the x-axis under T? The y-axis?

(d) Consider the line L given by $x + y = 1$. What is the image of L under T? Is it a circle, an ellipse, a hyperbole, or something else?

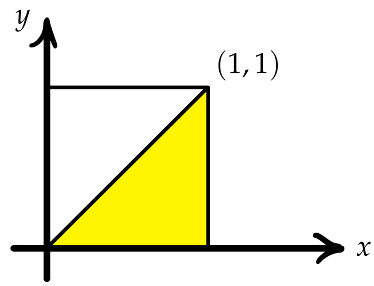
Hint: First, parameterize L by $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$ and then consider $\mathbf{f}(t) = T(\mathbf{r}(t))$.

(e) Consider the circle C given by $x^2 + y^2 = 1$. What is the image of C under T?

(f) Add $T(L)$, $T(C)$ and $T($ ☺ $)$ to your picture. Check your answer with the instructor.

Note: The transformation T is a particularly simple sort called a *linear transformation*.

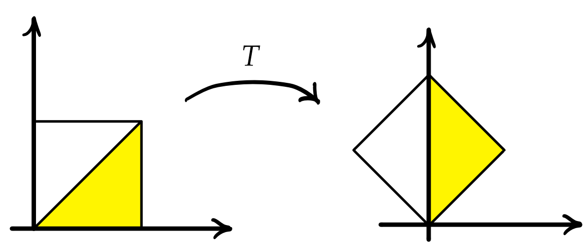
2. Consider the transformation $T(x, y) = \big(y, x(1 + y^2)\big)$. Draw the image of the picture below under T.



Hint: Parameterize each of the 5 line segments and proceed as in 1(d). To speed things, divide up the task.

Check your answer with the instructor.

3. In this problem, you'll construct a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates counter-clockwise about the origin by $\pi/4$, as shown below.



(a) Give a formula for T in terms of polar coordinates. That is, how does rotation affect r and θ?

(b) Write down T in terms of the usual rectangular (x, y) coordinates. Hint: first convert into polar, apply part (a) and then convert back into rectangular coordinates.

Check your answer with the instructor.