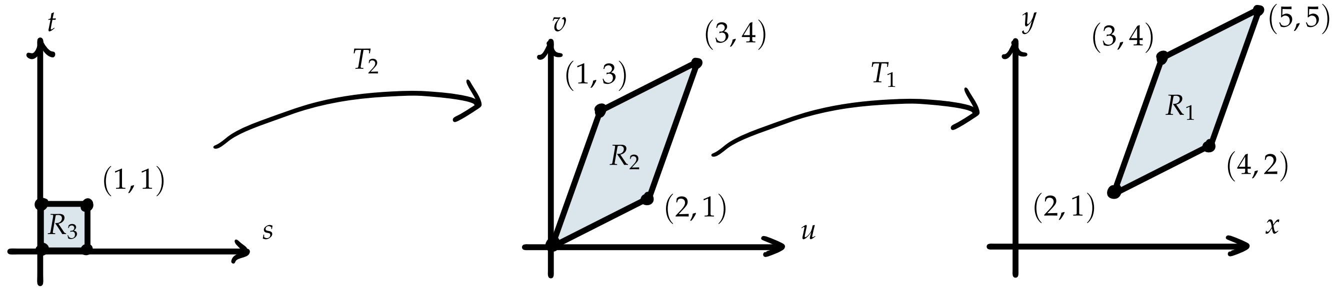
Thursday, November 3 $\quad ** \quad$ *Changing coordinates*

1. Consider the region R_1 in $\mathbb{R}^2$ shown below at right. In this problem, you will do a series of changes of coordinates to evaluate:

$$\iint_{R_1} x - 2y \, dA$$



   (a) A simple type of transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(s,t) = (s+a, t+b)$ for a fixed a and b. Find a translation T_1 such that $T_1(R_2) = R_1$.

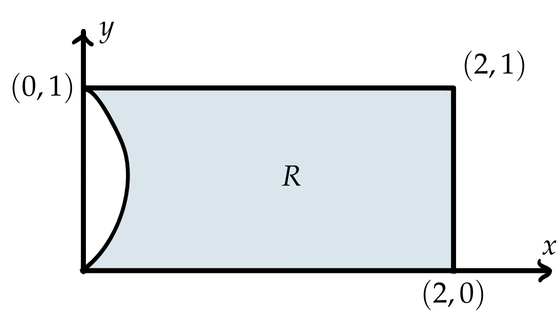
   (b) If T is a translation, what is its Jacobian matrix? How does it distort area?

   (c) Rewrite the original integral in terms of an integral over R_2.

   (d) Find a linear transformation $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square R_3 to R_2. Check your answer with the instructor.

   (e) Compute $\iint_{R_1} x - 2y \, dA$ by relating it to an integral over R_3 and evaluating that. Check your answer with the instructor.

2. Consider the region R shown below. Here the curved left side is given by $x = y - y^2$. In this problem, you will find a transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S = [0,1] \times [0,1]$ to R.



   (a) As a warm up, find a transformation that takes S to the rectangle $[0,2] \times [0,1]$ which contains R.

(b) Returning to the problem of finding T taking S to R, come up with formulas for $T(u, 0)$, $T(u, 1)$, $T(0, v)$, and $T(1, v)$. Hint: For three of these, use your answer in part (a).

(c) Now extend your answer in (b) to the needed transformation T. Hint: Try "filling in" between $T(0, v)$ and $T(1, v)$ with a straight line.

(d) Compute the area of R in two ways, once using T to change coordinates and once directly.

3. In order to do a change of coordinates in three variables, you need to compute a determinant of a 3×3 Jacobian matrix. In this problem, you will practice computing such determinants. Consider the 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) One method of computing 3×3 determinants is by considering the diagonals. The determinant of A can be computed as follows: add up the products along the three $\searrow$ diagonals and subtract off the products along the three $\swarrow$ diagonals. Find $\det B$.

(b) The "cofactor" method of computing determinants is as follows: pick a row of the matrix. For each entry on that row, multiply that entry by the determinant of the 2×2 matrix obtained by removing that row and column from the 3×3 matrix. The determinant is then the *alternating* sum of these products (alternating means that every other term has a negative sign). If you use the first or third row, the signs are $+ \; - \; +$, and if you use the second row, the signs are $- \; + \; -$.

For example, using the first row,

$$\det A = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

Compute $\det B$ by the method of cofactors once for each row. As you can see, it is usually a good idea to pick a row with the most 0's.