Name: NetID: UID: $\boxed{\quad /53}$

Instructions: Take care to note that problems are not weighted equally. Calculators, books, notes, and suchlike aids to gracious living are not permitted. **Show all your work** as credit will not be given for correct answers without proper justification, except for the "circle your answer" questions.

Important note: Several of the problems ask you to "completely setup but not evaluate" a certain integral. This means that all the limits of integration are specified, and the integrand is in terms of the final variables. For example, if S is a surface in $\mathbb{R}^3$, then an acceptable answer for setting up $\iint_S (x + y)^2 \, dA$ would be something like $\int_0^2 \int_0^{1-v} (u^2 + v \sin u) \, du \, dv$.

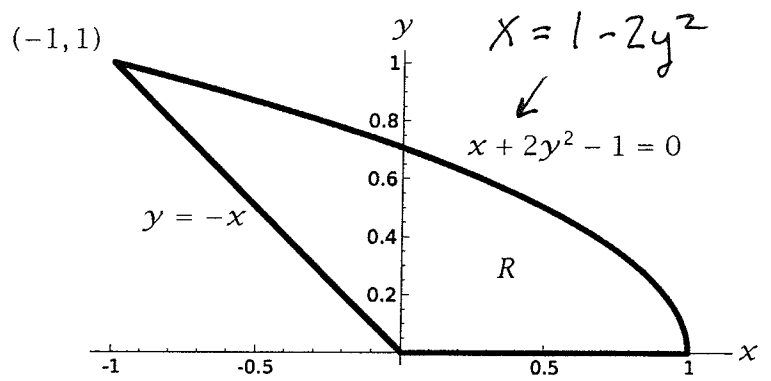**Scratch Space:** Below, and on back of last sheet.

Good luck!

Solution Key

Problem	Score	Out of
1	·	7
2		6
3		10
4		6
5		6
6		11
7		7
Total		53

1. Let R be the region shown at right. Evaluate $\iint_R y\, dA$. **(7 points)**

$(-1,1)$

$y = -x$

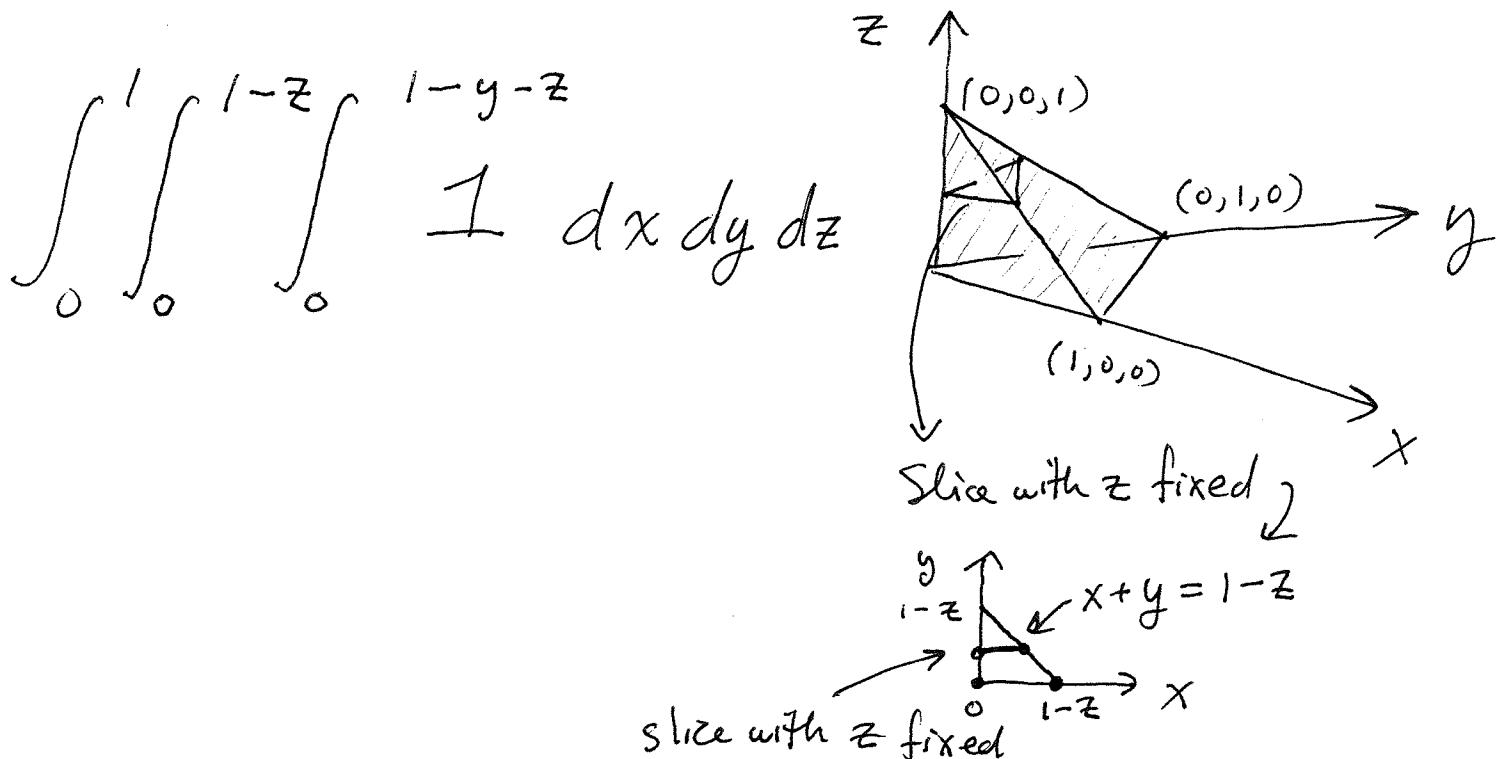$x = 1 - 2y^2$

$x + 2y^2 - 1 = 0$

R

$$\iint_R y\, dA$$

$$= \int_0^1 \int_{-y}^{1-2y^2} y\, dx\, dy = \int_0^1 yx \Big|_{x=-y}^{1-2y^2} dx$$

$$= \int_0^1 y(1-2y^2) - (y(-y))\, dy$$

$$= \int_0^1 y^2 - 2y^3 + y\, dy = \frac{y^3}{3} - \frac{2}{4}y^4 + \frac{y^2}{2} \Big|_{y=0}^{1}$$
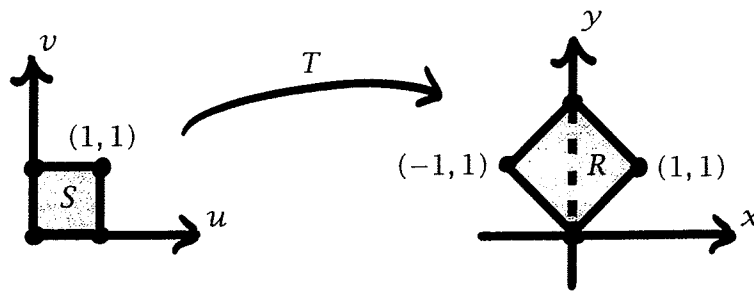
$$= \frac{1}{3}.$$

2. Consider the solid region E in the positive octant cut off by $x + y + z = 1$. Completely setup, but do not evaluate, a triple integral which computes the volume of E. **(6 points)**

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} 1\, dx\, dy\, dz$$

$(0,0,1)$

$(0,1,0)$

$(1,0,0)$

Slice with z fixed

$x + y = 1 - z$

slice with z fixed

3. Let R be the region shown at right.

(a) Find a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [0,1] \times [0,1]$ to R.   **(4 points)**

Using a linear transform $T(u,v) = (au+bv, cu+dv)$

$T(1,0) = (a, c) = (1,1) \Rightarrow \quad = c = 1.$

$T(0,1) = (b, d) = (-1,1) \Rightarrow b = -1, \; d = 1$

So: $T(u,v) = (\underbrace{u-v}_{x}, \; \underbrace{u+v}_{y}).$

(b) Use your change of coordinates to evaluate $\iint_R (x+y)^2 \, dA$ via an integral over S.   **(6 points)**

Emergency backup transformation: If you can't do (a), pretend you got the answer $T(u,v) = (uv, v)$ and do part (b) anyway.

$$\iint_R (x+y)^2 \, dA = \iint_S \left((u-v) + (u+v) \right)^2 \left| \det J \right| \, du \, dv$$

$$= \int_0^1 \int_0^1 4u^2 (2) \, du \, dv = 8 \int_0^1 \int_0^1 u^2 \, du \, dv$$

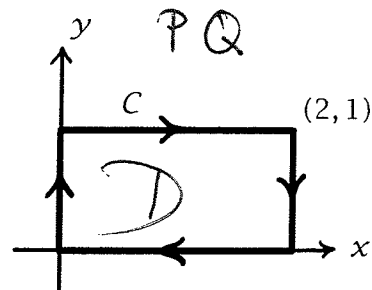$$= 8 \int_0^1 \left. \frac{u^3}{3} \right|_{u=0}^{1} dv$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \frac{8}{3} \int_0^1 dv = \boxed{\frac{8}{3}}.$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\det = 2.$

4. Let C be the oriented curve in $\mathbb{R}^2$ shown at right. For the vector field $F(x,y) = (x^3, x^2)$, use Green's theorem to evaluate $\int_C F \cdot dr$. **(6 points)**

$P \; Q$

y

C

$(2,1)$

D

x

$$\int_C F \cdot d\vec{r} = -\int_{\underbrace{-C}_{\substack{C \text{ reversed} \\ \text{so it's anti-} \\ \text{clockwise}}}} \vec{F} \cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

$$= -\int_0^1 \int_0^2 2x \, dx \, dy = -\int_0^1 x^2 \Big|_{x=0}^{2} \, dy$$

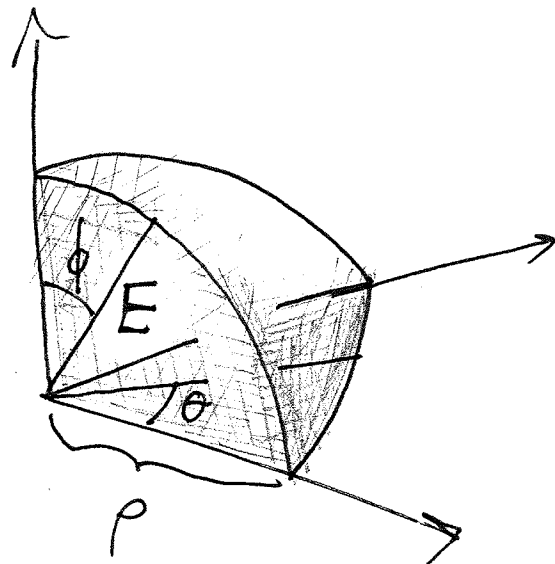$$= -\int_0^1 4 \, dy = \boxed{-4}$$

5. Let E be the portion of the positive octant which is inside the unit sphere. Use spherical coordinates to completely setup, but not evaluate, the integral $\iiint_E x + z \, dV$. **(6 points)**

$$\int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \left(\underbrace{\underset{\uparrow}{\quad} \right) \underbrace{\rho^2 \sin\phi \, d\theta \, d\phi \, d\rho}_{dV}$$

$\rho \sin\phi \cos\theta + \rho\cos\phi$

ϕ

E

θ

ρ
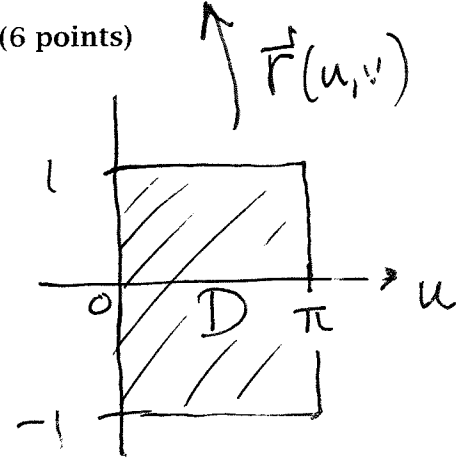
6. Let S be the surface in $\mathbb{R}^3$ parameterized by $r(u,v) = (v\cos u, v\sin u, u)$ for $0 \le u \le \pi$ and $-1 \le v \le 1$.

(a) Circle the correct picture of S.   **(2 points)**
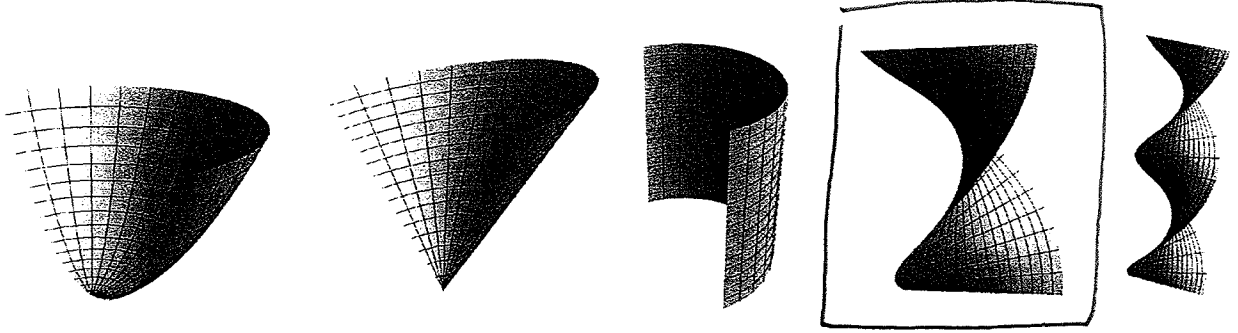
(b) Completely setup, but do not evaluate, the integral $\iint_S y \, dA$.   **(6 points)**

$$\iint_S y \, dA = \iint_D v\sin u \, |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$= \int_0^\pi \int_{-1}^1 v\sin u \sqrt{1+v^2} \; dv\, du$$

$$\vec{r}(u,v)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -v\sin u & v\cos u & 1 \\ \cos u & \sin u & 0 \end{vmatrix} = \left(-\sin u, \cos u, -v\right)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 u + \cos^2 u + v^2} = \sqrt{1+v^2}$$

(c) Find the equation for the tangent plane to S at the point $(0,0,\pi/2)$ in $\mathbb{R}^3$.   **(3 points)**

If $\vec{r}(u,v) = (0,0,\pi/2)$, this tells us that $u = \pi/2$.

As $\vec{r}(\pi/2, v) = (0, v, \pi/2)$, this implies that $v = 0$.

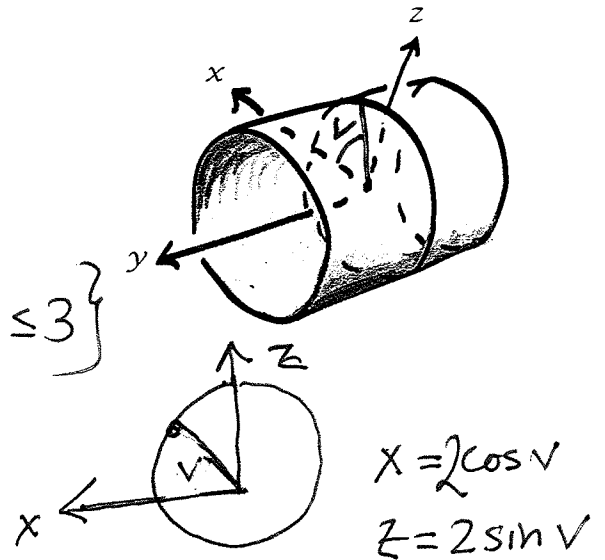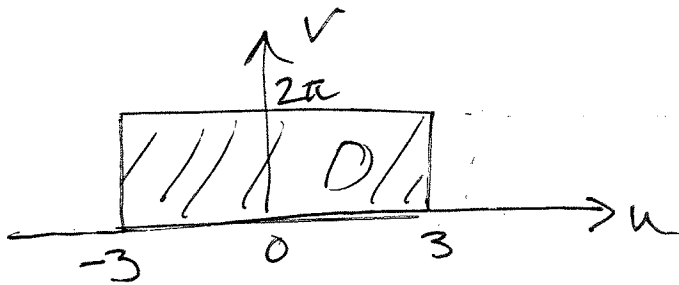Normal vector to plane is $\vec{r}_u \times \vec{r}_v = (-1, 0, 0)$

at $(u,v) = (\pi/2, 0)$.   Hence the plane is $-1(x-0) = 0$,

i.e. the yz-plane.

7. For each surface S below, give a parameterization $\mathbf{r}: D \to S$. Be sure to explicitly specify the domain D.

(a) The portion of the cylinder $x^2 + z^2 = 4$ where $-3 \leq y \leq 3$. **(4 points)**

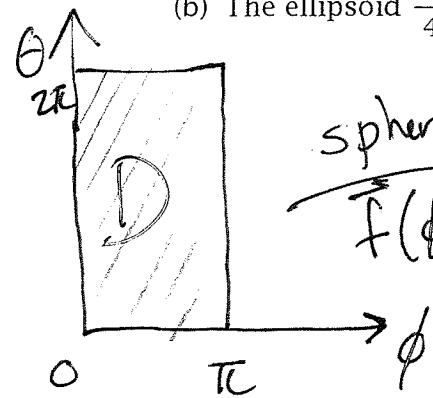Parameters $y = u$

Angle $= v$

$D = \{0 \leq v \leq 2\pi, -3 \leq y \leq 3\}$
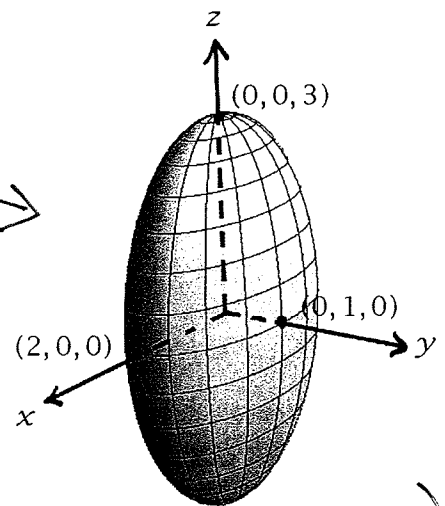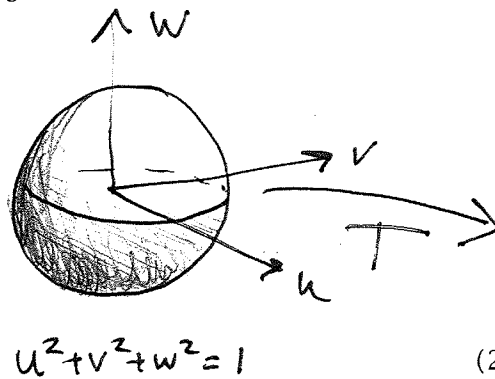
$X = 2\cos v$

$Z = 2\sin v$

$\vec{r}(u,v) = (2\cos v, u, 2\sin v)$

(b) The ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 1$. **(3 points)**

spherical $\vec{f}(\phi, \theta)$

$u^2 + v^2 + w^2 = 1$

$(0,0,3)$

$(0,1,0)$

$(2,0,0)$

So:

$D = \{0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi\}$ $T(u,v,w) = (2u, v, 3w)$

and

$\vec{r}(\phi, \theta) = T(\vec{f}(\phi, \theta)) = (2\sin\phi\cos\theta, \sin\phi\sin\theta, 3\cos\phi)$