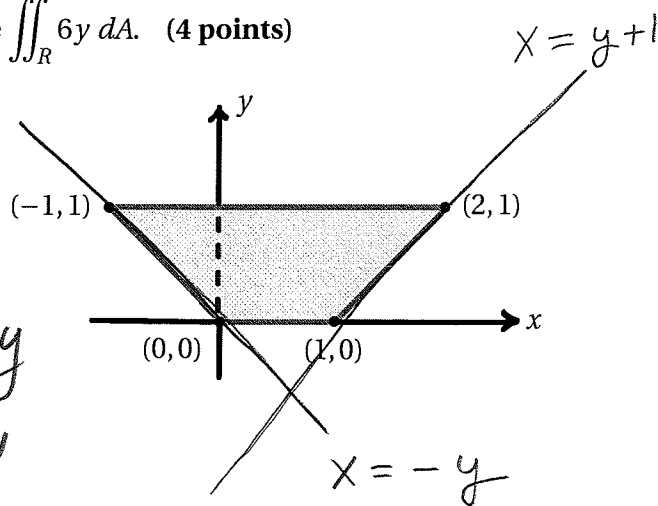


1. Let R be the region of integration pictured below right. Evaluate $\iint_R 6y \, dA$. (4 points)

$$\begin{aligned} & \int_0^1 \int_{-y}^{y+1} 6y \, dx \, dy \\ &= \int_0^1 6yx \Big|_{x=-y}^{y+1} dy = \int_0^1 6y(2y+1) dy \\ &= \int_0^1 12y^2 + 6y \, dy = 4y^3 + 3y^2 \Big|_{y=0}^{y=1} \\ &= 7 \end{aligned}$$



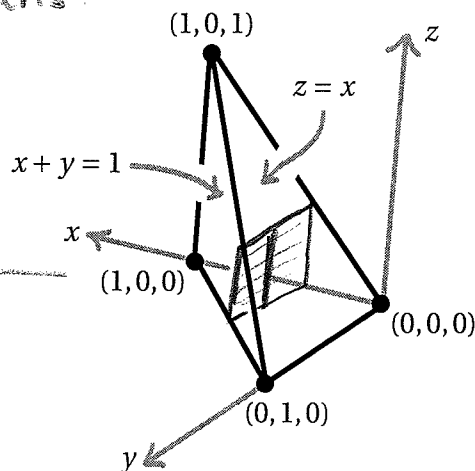
$$\boxed{\iint_R 6y \, dA = 7}$$

2. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the tetrahedron shown at right. (5 points)

Two common correct ans:

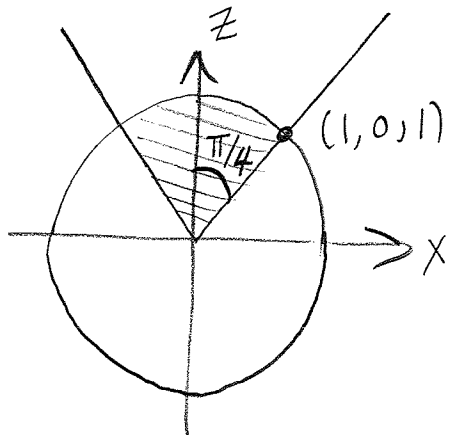
a) $\int_0^1 \int_0^{1-x} \int_0^x 1 \, dz \, dy \, dx$

b) $\int_0^1 \int_0^{1-y} \int_0^x 1 \, dz \, dx \, dy$



3. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$. (5 points)

Cross section:



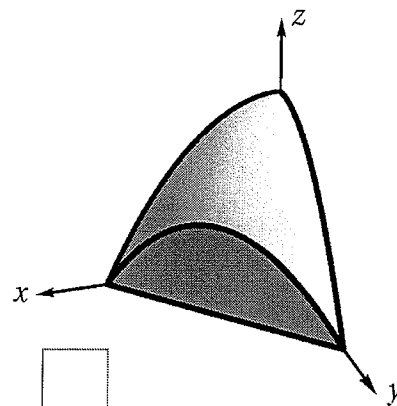
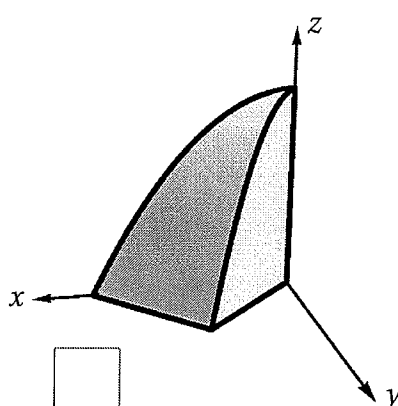
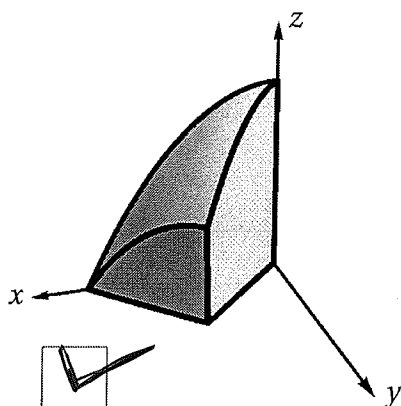
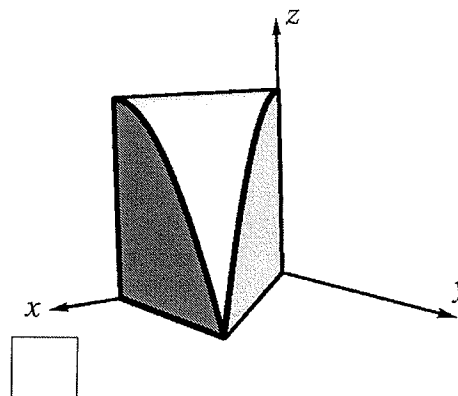
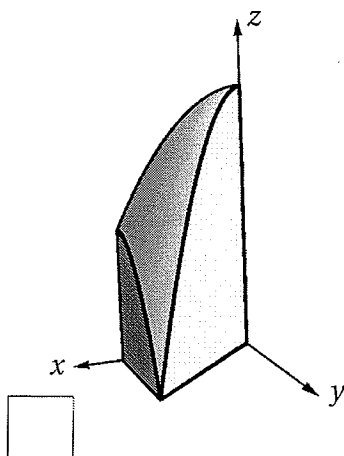
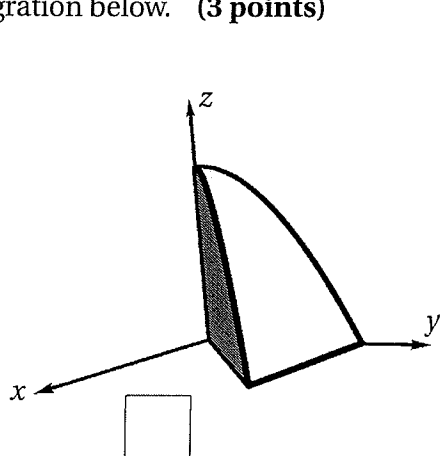
3D:



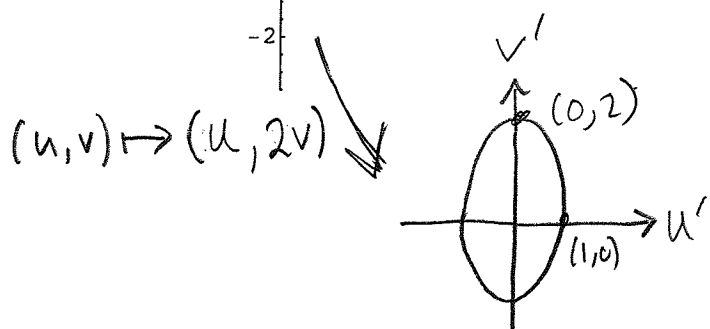
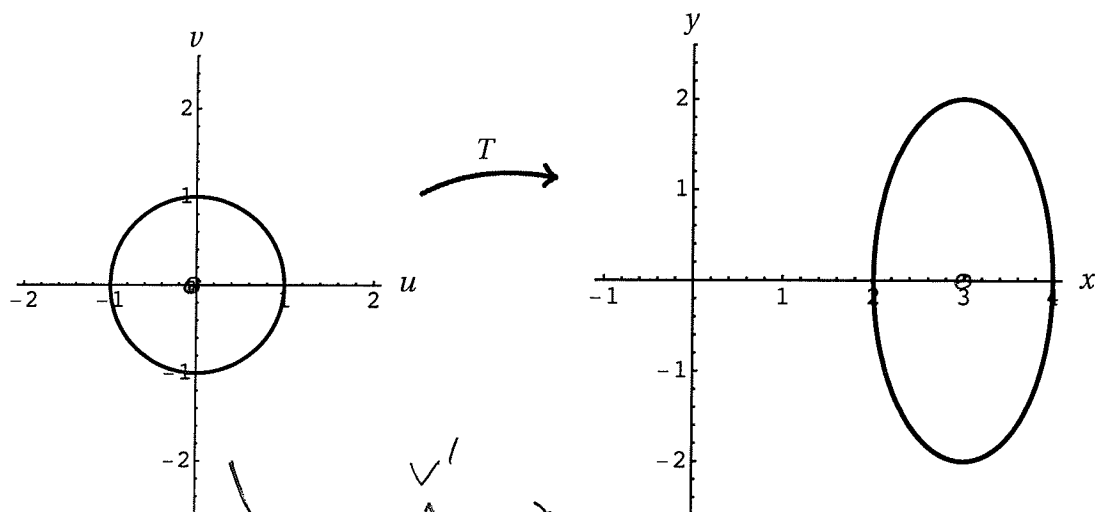
Using spherical coordinates.

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 1 \, dV$$

4. Consider the triple integral $\int_0^{1/2} \int_y^{1-y} \int_0^{1-x^2-y^2} f(x, y, z) \, dz \, dx \, dy$. Mark the corresponding region of integration below. (3 points)



5. Find a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which takes the unit circle to the ellipse given by $(x-3)^2 + \frac{y^2}{4} = 1$ as shown. (3 points)

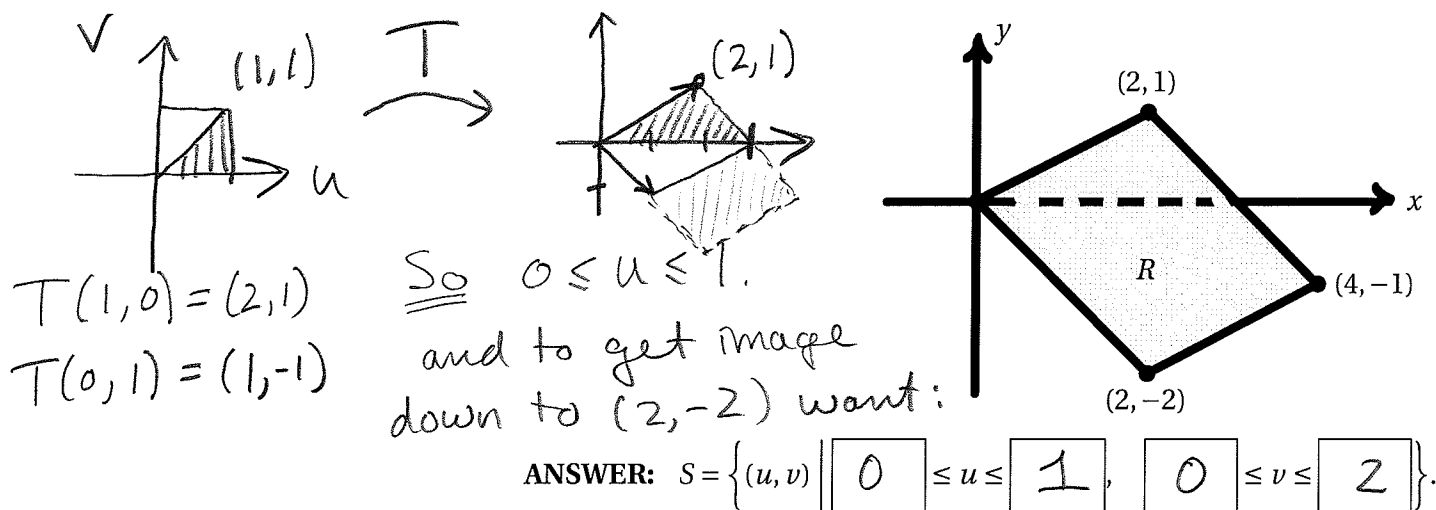


Handwritten transformation: $(u', v') \mapsto (u' + 3, v')$

Final transformation: $T(u, v) = (u + 3, 2v)$

6. Let R be the region in the xy -plane depicted below right. Let $T(u, v) = (2u + v, u - v)$.

- (a) Find a rectangle S in the uv -plane whose image under T (that is, the collection of points $T(u, v)$ for all choices of (u, v) in S) is exactly R . (3 points)



- (b) Set up, but DO NOT EVALUATE, the integral $\iint_R \cos(x) dA$ as an integral in the (u, v) -coordinates. If you can't do part (a), leave the limits of integration blank. (5 points)

$$\iint_R \cos x dA = \int_0^2 \int_0^1 \cos(2u+v) | \det J | du dv$$

$$= \int_0^2 \int_0^1 3 \cos(2u+v) du dv$$

$$J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det J = -2 - 1 = -3$$

$$\iint_R \cos(x) dA = \int_0^2 \int_0^1 3 \cos(2u+v) du dv$$

7. Let $\mathbf{F}(x, y) = \langle x^2, x^2 \cos(y) \rangle$. Then $\iint_R \left[\frac{\partial}{\partial x} (x^2 \cos(y)) - \frac{\partial}{\partial y} (x^2) \right] dA = 0$ where R is the region shown below.

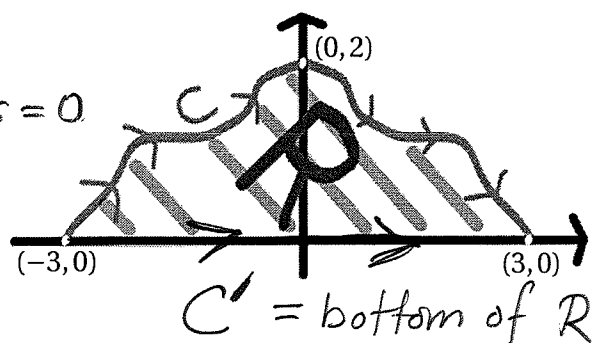
Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the pictured curve that goes from $(-3, 0)$ to $(3, 0)$ via $(0, 2)$. (3 points)

By Green, we have $\int_{C-C'} \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = 0$

Thus $\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$

$= \int_{C'} x^2 dx + \int_{C'} x^2 \cos y dy$ ← always = 0

$= \int_{-3}^3 x^2 dx = \frac{x^3}{3} \Big|_{-3}^3 = 18$



8. Consider the region D in the plane bounded by the curve C as shown at right. For each part, circle the best answer. (1 point each)

(a) For $\mathbf{F}(x, y) = \langle x+1, y^2 \rangle$, the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is

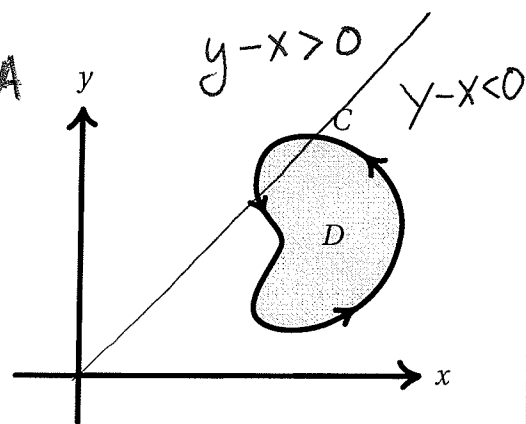
negative ☒ zero ☐ positive

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\ &= \iint_R 0 dA = 0 \end{aligned}$$

(b) The integral $\int_C (-y dx + 2 dy)$ is

negative ☐ zero ☒ positive

$$\begin{aligned} \int_C (-y dx + 2 dy) &= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_R 1 dA \\ &= \text{Area}(R) \end{aligned}$$



(c) The integral $\iint_D (y-x) dA$ is

☒ negative ☐ zero ☐ positive

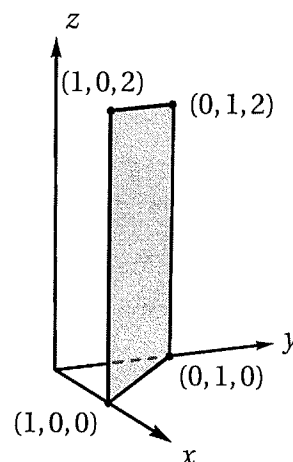
9. For each surface S below, give a parameterization $\mathbf{r}: D \rightarrow S$. Be sure to explicitly specify the domain D and call your parameters u and v .

(a) The rectangle in $\mathbb{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(1, 0, 2)$, $(0, 1, 2)$. (3 points)

Can use coordinates x and z as parameters, call them u and v . Since this rect. is contained in the line $x+y=1$, get $y=1-x=1-u$

$$D = \{ 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 2 \}$$

$$\mathbf{r}(u, v) = \langle u, 1-u, v \rangle$$

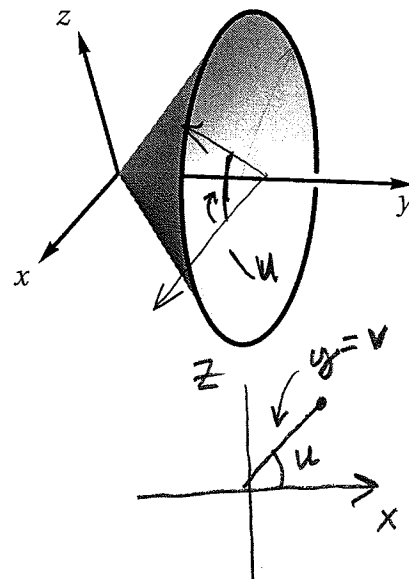


(b) The portion of cone $y = \sqrt{x^2 + z^2}$ for $0 \leq y \leq 1$ which is shown at right. (4 points)

Can use $v=y$ and the angle u shown as the parameters. Also, the radius of the circle with y fixed is just y , and hence:

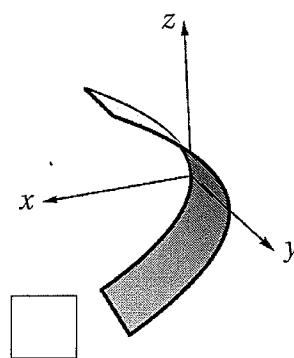
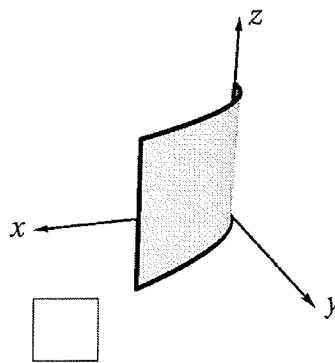
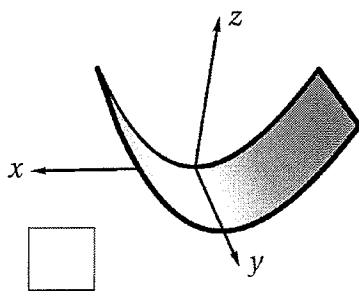
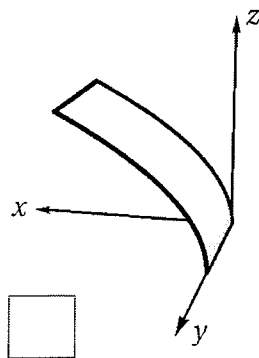
$$D = \{ 0 \leq u \leq 2\pi \text{ and } 0 \leq v \leq 1 \}$$

$$\mathbf{r}(u, v) = \langle v \cos u, v, v \sin u \rangle$$



10. Consider the surface S parameterized by $\mathbf{r}(u, v) = (v^2, u, v)$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

(a) Mark the correct picture of S below. (2 points)



(b) Evaluate the integral $\iint_S z \, dA$. (6 points)

$$\iint_S z \, dA = \int_0^1 \int_0^1 v |\vec{r}_u \times \vec{r}_v| \, du \, dv = \int_0^1 \int_0^1 v \sqrt{1+4v^2} \, du \, dv$$

$$= \int_0^1 v \sqrt{1+4v^2} \, dv = \int_1^5 \frac{1}{8} \sqrt{w} \, dw$$

$$dw = 8v \, dv$$

$$= \frac{1}{12} w^{3/2} \Big|_{w=1}^{w=5} = \frac{1}{12} (5^{3/2} - 1)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = (1, 0, -2v)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+4v^2}$$

$$\boxed{\iint_S z \, dA = \frac{1}{12} (5^{3/2} - 1)}$$

11. Consider the solid described as follows using cylindrical coordinates: E is the region inside the paraboloid $z = 1 - r^2$ and where $0 \leq \theta \leq \pi$ and $z \geq 0$. Choose one double integral and one triple integral below that compute the volume of E . (1 point each)

☐ $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} 1 \, dz \, dx \, dy$

☒ $\int_0^1 \int_0^{\sqrt{1-z}} 2\sqrt{1-z-y^2} \, dy \, dz$

☒ $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} 1 \, dy \, dx \, dz$

☐ $\int_0^1 \int_0^{\sqrt{1-z^2}} 2\sqrt{1-y^2-z^2} \, dy \, dz$

☐ $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-\sqrt{x^2+y^2}} 1 \, dz \, dx \, dy$

☐ $\int_0^1 \int_0^{1-z} 2\sqrt{(1-z)^2 - y^2} \, dy \, dz$