MA 114 Worksheet # 10:
Taylor polynomials and volumes of solids with known cross sections

Recommendation: Drawing a picture of the solids may be helpful during this worksheet.

1. What is $T_3(x)$ centered at $a = 3$ for a function $f(x)$ where $f(3) = 9$, $f'(3) = 8$, $f''(3) = 4$, and $f'''(3) = 12$?

2. Calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x = a$ for the given function and value of a.

   (a) $f(x) = \tan x$, $a = \frac{\pi}{4}$

   (b) $f(x) = x^2 e^{-x}$, $a = 1$

   (c) $f(x) = \dfrac{\ln x}{x}$, $a = 1$

3. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of $|f(1.1) - T_2(1.1)|$. Show that we can take $K = e^{1.1}$.

4. (a) Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_k(x)$ for $k = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.

   (b) Let $T_n(x)$ be the n^{th} Taylor polynomial at $x = a$ for a polynomial $f(x)$ of degree n. Based on part (a), guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

5. Conceptual Understanding: If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

6. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4. Use similar triangles to find the area of a horizontal cross section at a height y. Using this area, calculate V by integrating the cross-sectional area.

7. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval $[0, l]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

8. Calculate the volume of the following solid. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

9. The base of a certain solid is the triangle with vertices at $(-10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

10. As viewed from above, a swimming pool has the shape of the ellipse $\dfrac{x^2}{2500} + \dfrac{y^2}{1600} = 1$. The cross sections perpendicular to the ground and parallel to the y-axis are squares. Find the total volume of the pool.

11. Calculate the volume of the following solid. The base is a circle of radius r centered at the origin. The cross sections perpendicular to the x-axis are squares.

12. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \le y \le 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.