Math 114 Worksheet # 1: Integration by Parts

1. Use the product rule to find $(u(x)v(x))'$. Next use this result to prove integration by parts, namely that $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)\,dx$.

2. Which of the following integrals should be solved using substitution and which should be solved using integration by parts?

   (a) $\int x\cos(x^2)\,dx$,

   (b) $\int e^x \sin(x)\,dx$,

   (c) $\int \frac{\ln(\arctan(x))}{1+x^2}dx$,

   (d) $\int xe^{x^2}\,dx$

   Using these examples, try and formulate a general rule for when integration by parts should be used as opposed to substitution.

3. Solve the following integrals using integration by parts:

   (a) $\int x^2\sin(x)\,dx$,

   (b) $\int (2x+1)e^x\,dx$,

   (c) $\int x\sin(3-x)\,dx$,

   (d) $\int 2x\arctan(x)\,dx$,

   (e) $\int \ln(x)\,dx$

4. Prove the reduction formula $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$. Use this to evaluate $\int x^3 e^x dx$.

5. Let $f(x)$ be a twice differentiable function with $f(0) = 6$, $f(1) = 5$, and $f'(1) = 2$. Evaluate $\int_0^1 xf''(x)\,dx$.

6. Evaluate $\int \sin(x)\cos(x)\,dx$ by four methods

   (a) the substitution $u = \cos(x)$,

   (b) the substitution $u = \sin(x)$,

   (c) the identity $\sin(2x) = 2\sin(x)\cos(x)$,

   (d) integration by parts.

MA 114 Worksheet # 2: Improper Integrals

1. For each of the following, determine if the integral is proper or improper. If it is improper, explain why. Do ***not*** evaluate any of the integrals.

   (a) $\displaystyle\int_0^2 \frac{x}{x^2-5x+6}\,dx$

   (b) $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$

   (c) $\displaystyle\int_1^2 \ln(x-1)\,dx$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$

   (e) $\displaystyle\int_0^{\pi/2} \sec x\,dx$

2. For the integrals below, determine if the integral is convergent or divergent. Evaluate the convergent integrals.

   (a) $\displaystyle\int_{-\infty}^{0} \frac{1}{2x-1}\,dx$

   (b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

   (c) $\displaystyle\int_0^2 \frac{x-3}{2x-3}\,dx$
   (write the numerator as $\frac{1}{2}(2x-3)-\frac{3}{2}$)

   (d) $\displaystyle\int_0^{\infty} \sin\theta\,d\theta$

3. Consider the improper integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx.$$

   Integrate using the generic parameter p to prove the integral converges for $p>1$ and diverges for $p\le 1$. You will have to distinguish between the cases when $p=1$ and $p\neq 1$ when you integrate.

4. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.

   (a) $\displaystyle\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$

   (b) $\displaystyle\int_1^{\infty} \frac{x+1}{\sqrt{x^6+x}}\,dx$

5. Explain why the following computation is wrong and determine the correct answer. (Try sketching or graphing the integrand to see where the problem lies.)

$$\begin{aligned}\int_2^{10} \frac{1}{2x-8}dx &= \frac{1}{2}\int_{-4}^{12} \frac{1}{u}\,du\\ &= \frac{1}{2}\ln|x|\Big|_{-4}^{12}\\ &= \frac{1}{2}(\ln 12-\ln 4)\end{aligned}$$

   where we used the substitution

$$\begin{cases} u(x)=2x-8\\ u(2)=-4,\ u(10)=12\\ \dfrac{du}{dx}=2\end{cases}$$

MA 114 Worksheet # 3: Sequences

1. Write the first four terms of the sequences with the following general terms:

   (a) $\dfrac{n!}{2^n}$

   (b) $\dfrac{n}{n+1}$

   (c) $(-1)^{n+1}$

2. Find a formula for the nth term of the sequence $\left\{\dfrac{1}{1}, -\dfrac{1}{8}, \dfrac{1}{27}, -\dfrac{1}{64}, \ldots\right\}$.

3. Conceptual Understanding:

   (a) What is a sequence?

   (b) What does it mean to say that a sequence is bounded?

   (c) What does it mean to say that a sequence is defined recursively?

   (d) What does it mean to say that a sequence converges?

4. Let $a_0 = 0$ and $a_1 = 1$. Write out the first five terms of $\{a_n\}$ where a_n is recursively defined as $a_{n+1} = 3a_{n-1} + {a_n}^2$.

5. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, produce a counterexample.

6. Show that the sequence with general term $a_n = \dfrac{3n^2}{n^2+2}$ is increasing. Find and upper bound. Does $\{a_n\}$ converge?

7. Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

   (a) $a_n = 1.01^n$.

   (b) $b_n = \frac{3n^2+n+1}{2n^2-3}$.

   (c) $c_n = e^{1-n^2}$.

MA 114 Worksheet # 4: Summing an Infinite Series

1. (Review) Compute the following sums

   (a) $\sum_{n=1}^{5} 3n$

   (b) $\sum_{k=3}^{6} \left(\sin\left(\frac{\pi}{2} + \pi k\right) + 2k \right)$

2. Conceptual Understanding:

   (a) What is a series?

   (b) What is the difference between a sequence and a series?

   (c) What does it mean that a series converges?

3. Write the following in summation notation:

   (a) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$

   (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

4. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series $S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$.

5. Use Theorem 3 of 10.2 (Divergence Test) to prove that the following two series diverge:

   (a) $\sum_{n=1}^{\infty} \frac{n}{10n+12}$

   (b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

6. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:

   (a) $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \dots$.

   (b) $\sum_{n=0}^{\infty} \left(\frac{\pi}{e} \right)^n$.

   (c) $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \dots$.

   (d) $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$.

MA 114 Worksheet # 5: Series with Positive Terms

1. Use the Integral Test to determine if the following series converge or diverge:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$

2. Show that the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise by Integral Test.

3. Use the Comparison Test (or Limit Comparison Test) to determine whether the infinite series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2+5^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^4}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$

MA 114 Worksheet # 6: Absolute and Conditional Convergence

1. Conceptual understanding:

   (a) Let $a_n = \dfrac{n}{3n+1}$. Does $\{a_n\}$ converge? Does $\sum_{n=1}^{\infty} a_n$ converge?

   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} a_n = 0$.

   (c) Does there exist a convergent series $\sum_{n=1}^{\infty} a_n$ which satisfies $\lim_{n\to\infty} a_n \neq 0$? Explain.

   (d) When does a series converge absolutely? When does a series converge conditionally?

   (e) State the Leibniz test for alternating series.

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

   (a) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{\sqrt[5]{n}}$

   (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n}}{\ln(n+1)}$

   (c) $\sum_{n=1}^{\infty} 13\cos(5)^{n-1}$

3. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.

   (a) To prove that the series $\sum_{n=1}^{\infty} a_n$ converges you should compute the limit $\lim_{n\to\infty} a_n$. If this limit is 0 then the series converges.

   (b) One way to prove that a series is convergent is to prove that it is absolutely convergent.

   (c) An infinite series converges when the limit of the sequence of partial sums converges.

MA 114 Worksheet # 7:
Ratio and Root Test & Power Series

1. (a) State the Root Test.
   (b) State the Ratio Test.

2. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^3+2n}{4n^3+1}\right)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

3. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.

   (a) To apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If this limit is less than 1 then the series converges absolutely.

   (b) To apply the Root Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is 1 or larger then the series diverges.

4. Give the definition of the radius of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$

5. Find the radius and interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$.

6. Find the radius and interval of convergence for $\displaystyle 4\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$.

7. Find the radius and interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$.

8. Find the radius and interval of convergence for $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^n$.

MA 114 Worksheet # 8: Review for Exam 1

1. Integration by Parts:

   (a) $\int x^2 \cos(x)\, dx$

   (b) $\int 2x \arctan(x)\, dx$

2. Improper Integrals:

   (a) Evaluate $\int_{-1}^{\infty} e^{-x}\, dx$

   (b) Evaluate $\int_{1}^{2} \frac{dx}{x \ln(x)}$.

   (c) Evaluate $\int_{3}^{6} \frac{x}{\sqrt{x-3}}\, dx$.

3. Determine the limit of the sequence or state that the sequence diverges.

   (a) $a_n = \sqrt{n+3} - \sqrt{n}$

   (b) $b_n = \frac{\cos(n)}{n}$

   (c) $c_n = \frac{e^n + (-3)^n}{5^n}$

   (d) $d_n = n^{1/n}$
   [HINT: Verify that $x^{1/x} = e^{\frac{\ln(x)}{x}}$ and use l'Hospital's Rule.]

4. Determine whether or not $\sum_{n=2}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n^2}}\right)$ converges.

5. Evaluate $\sum_{n=3}^{\infty} \frac{1}{n(n+3)}$.
   [HINT: Use $\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$.]

6. Find a value of N such that S_N approximates the series with an error of at most 10^{-5} where $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$

7. Determine convergence or divergence.

   (a) $\sum_{n=0}^{\infty} 5^{-n}$.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$.

   (c) $\sum_{n=1}^{\infty} n e^{-n^2}$.

   (d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$.

   (e) $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$.

8. Determine the radius of convergence of the following series.

   (a) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

   (b) $\sum_{n=2}^{\infty} \frac{(2x-3)^n}{n \ln(n)}$

MA 114 Worksheet # 9: Power Series & Taylor Series

1. Use term by term integration and the fact that $\int \frac{1}{1+x^2}\,dx = \arctan(x)$ to derive a power series centered at $x = 0$ for the arctangent function.
   [HINT: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.]

2. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\int \frac{dx}{1+x} = \ln(1+x)$. You will again want to manipulate the fraction $\frac{1}{1+x} = \frac{1}{1-(-x)}$ as above.

3. Write $(1+x^2)^{-2}$ as a power series.
   [HINT: Use term by term differentiation.]

4. Find the terms through degree 3 of the Maclaurin series of $f(x)$.

   (a) $f(x) = (1+x)^{1/4}$.
   (b) $f(x) = e^{\sin(x)}$.

5. Find the Taylor series centered at c and find the interval on which the expansion converges to f.

   (a) $f(x) = \frac{1}{x}$ at $c = 1$.
   (b) $f(x) = e^{3x}$ at $c = -1$.
   (c) $f(x) = x^3 + 3x - 1$ at $c = 0$.
   (d) $f(x) = x^3 + 3x - 1$ at $c = 2$.

MA 114 Worksheet # 10: Taylor Series & Taylor Polynomials

1. Find a power series representation for

   (a) $f(x) = x\cos(x^2)$.

   (b) $g(x) = (1+x)e^{-x}$.

2. Show that $\lim_{x\to 0} \frac{e^x - \cos(x)}{\sin(x)} = 1$ using power series. Verify your answer with l'Hospital's Rule.
   [HINT: Write out the power series for each term and factor out the lowest power of x from the numerator and the denominator, and then consider the limit.]

3. What is $T_3(x)$ centered at $a = 3$ for a function $f(x)$ where $f(3) = 9$, $f'(3) = 8$, $f''(3) = 4$, and $f'''(3) = 12$?

4. Calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x = a$ for the given function and value of a.

   (a) $f(x) = \tan x$, $a = \frac{\pi}{4}$

   (b) $f(x) = x^2 e^{-x}$, $a = 1$

   (c) $f(x) = \dfrac{\ln x}{x}$, $a = 1$

5. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of $|f(1.1) - T_2(1.1)|$. Show that we can take $K = e^{1.1}$.

6. (a) Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_k(x)$ for $k = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.

   (b) Let $T_n(x)$ be the n^{th} Taylor polynomial at $x = a$ for a polynomial $f(x)$ of degree n. Based on part (a), guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

MA 114 Worksheet # 11: Volumes of solids with known cross sections

Recommendation: Drawing a picture of the solids may be helpful during this worksheet.

1. Conceptual Understanding: If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4. Use similar triangles to find the area of a horizontal cross section at a height y. Using this area, calculate V by integrating the cross-sectional area.

3. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval $[0, l]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

4. Calculate the volume of the following solid. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

5. The base of a certain solid is the triangle with vertices at $(-10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

6. As viewed from above, a swimming pool has the shape of the ellipse $\dfrac{x^2}{2500} + \dfrac{y^2}{1600} = 1$. The cross sections perpendicular to the ground and parallel to the y-axis are squares. Find the total volume of the pool.

7. Calculate the volume of the following solid. The base is a circle of radius r centered at the origin. The cross sections perpendicular to the x-axis are squares.

8. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \leq y \leq 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.

Worksheet #12:
Density and Average Value & Volumes of Revolution (Disk Method)

1. Conceptual Understanding:

   (a) If the linear mass density of a rod at position x is given by the function $\rho(x)$, what integral should be evaluated to find the mass of the rod between points a and b?

   (b) If the radial mass density of a disk centered at the origin is given by the function $\rho(r)$, where r is the distance from the center point, what integral should be evaluated to find the mass of a disk of radius R?

   (c) Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the total mass of a 1-meter rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 2$.

3. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$

   (b) $f(x) = x^3,\ [-1, 1]$

   (c) $f(x) = \cos(x),\ \left[0, \frac{\pi}{6}\right]$

   (d) $f(x) = \dfrac{1}{x^2+1},\ [-1, 1]$

   (e) $f(x) = \dfrac{\sin(\pi/x)}{x^2},\ [1, 2]$

   (f) $f(x) = e^{-nx},\ [-1, 1]$

   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$

   (h) $f(x) = x^n$ for $n \ge 0$, $[0, 1]$

4. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1+r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5 km radius of the clearing.

5. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

6. Find the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x^5}$, $y = 0$, $x = 1$, and $x = 6$, about the x-axis.

7. Find the volume of the solid obtained by rotating the region bounded by $f(x) = \sin(x)$ and the x-axis over the interval $[0, \pi]$ about the x-axis.
   [Hint: You may use the trig identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ in order to evaluate the integral.]

8. Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $x = 0$, $y = 1$, $x = y^{11}$, about the line $y = 1$.

9. For each of the following, use disks or washers to find the integral expression for the volume of the region.

   (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.

   (b) R is the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the y-axis.

   (c) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.

   (d) Between the regions in part (a) and part (c), which volume is bigger? Why? First argue without computing the integrals, then also evaluate the integrals to check your answer.

   (e) R is the region bounded by $y = e^{-x}$, $y = 1$ and $x = 2$; about the line $y = 2$.

(f) R is the region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

10. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. $y = 0$, $y = \cos(2x)$, $x = \frac{\pi}{2}$, $x = 0$ about the line $y = -6$.

11. Find the volume of the cone obtained by rotating the region in the first quadrant under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

12. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the y-axis. Soda is extracted from the glass through a straw at the rate of 1/2 cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

13. The torus is the solid obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]

MA 114 Worksheet #13: Volumes of Revolution (Shell Method)

1. Conceptual understanding of disk and shell method:

   (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y-axis using the method of cylindrical shells.

   (b) If you use the disk method to compute the same volume, are you integrating with respect to x or y? Why?

2. Sketch the enclosed region and use the Shell Method to calculate the volume of rotation about the y-axis.

   (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$

   (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$

3. Sketch the enclosed region and use the Shell Method to calculate the volume of the solid when rotated about the x-axis.

   (a) $x = \frac{1}{4}y + 1$, $x = 3 - \frac{1}{4}y$, $y = 0$

   (b) $x = y(4 - y)$, $x = 0$

4. Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \leq x \leq 2$ about:

   (a) the x-axis

   (b) the y-axis

5. Use the Shell method to find the volume obtained by rotating the region bounded by $y = x^2 + 2$, $y = 6$, $x = 0$, and $x = 2$ about the following axes:

   (a) $x = 2$

   (b) $x = -3$

6. Find the volume of the solid obtained by rotating the following region about the y-axis.

   (a) The region bounded by $f(x) = e^x$ and the x-axis from $0 \leq x \leq 2$.

   (b) The region bounded by $f(x) = \sin(x)$ and the x-axis from $0 \leq x \leq \pi$.

   (c) The region bounded $f(x) = \ln(x)$ and the x-axis from $1 \leq x \leq 3$.

Math 114 Worksheet # 14:
Work & Trigonometric Integrals

1. Conceptual Understanding:
   (a) Define and describe work. What are its units? What is the difference between work and force?
   (b) Determine the work done in lifting a 1 kg weight through a distance of 1 m near the surface of the earth, maintaining a constant velocity.
   (c) How much work is done in lifting a 1 kg weight up 1 m at a constant velocity and then lowering it back 1 m at a constant velocity?
2. Determine the work done in lifting a 500 kg elevator 1000 m to the top floor of a building. How much work is done lowering a 500 kg elevator 1000 m from the top floor of a building to the ground floor? How much work is done making the round trip?
3. A force of 50 N will stretch a spring from its natural length of 5 cm to 15 cm. How much work will be done in stretching the spring from 15 cm to 30 cm?
4. Calculate the work against gravity required to build a right circular cone of height 4 m and base radius 1.2 m out of a lightweight material of density 600 kg/m^3.
5. Consider a rectangular tank of water that is 5 meters tall and has a base of size 8×4 meters. It has a spout on its top surface. Calculate the work required to pump all of the water out of the tank. Dimensions are in meters, and the density of water is 1000 kg/m^3.
6. Evaluate the following integrals.
   (a) $\int \cos^2(x)\,dx$.
   (b) $\int_0^{\pi/2} \sin^2(x)\cos^2(x)\,dx$.
   (c) $\int \sin^3(x)\cos^2(x)\,dx$.
   (d) $\int x^2\cos(x)\,dx$.
   (e) $\int e^x\cos(x)\,dx$.

Math 114 Worksheet # 15: Trigonometric Integrals

1. Evaluate the following integrals.

   (a) $\int \tan^2(x)\,dx$

   (b) $\int \frac{\sin(\varphi)}{\cos^3(\varphi)}\,d\varphi$

   (c) $\int \tan^5(x)\sec^3(x)\,dx$

   (d) $\int \sin(8x)\cos(5x)\,dx$

   (e) $\int \frac{1-\tan^2(x)}{\sec^2(x)}\,dx$

   (f) $\int x\sec^2(x^2)\tan^4(x^2)\,dx$

   (g) $\int_{-\pi/4}^{\pi/4} \tan^3(x)\,dx$

   (h) $\int_{\pi/4}^{\pi/2} \cot^3(x)\,dx$

2. Prove that for any two integers m and n

$$\frac{1}{\pi}\int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}.$$

MA 114 Worksheet # 16: Review for Exam 2

1. Power, Maclaurin, and Taylor Series
   (a) Find the Maclaurin series for $\dfrac{x^2}{1+x}$.
   (b) Find the Taylor series for $\cos x$ about $a = \pi/2$.
   (c) Find the Taylor series centered at $c = 0$ of $\dfrac{2}{4-3x}$ and determine its radius of convergence.
   (d) Find the Taylor series centered at zero of the function $f(x) = \ln(x+5)$.
   (e) Find the Taylor series centered at zero of the function $g(x) = x^3 \ln(x^2+5)$.

2. Compute $T_3(x)$, the Taylor polynomial of the third order centered at $x = 0$, for $f(x) = \cos(x/\pi)$.

3. Compute $T_n(x)$, the Taylor polynomial of the nth order centered at $x = 0$, for $f(x) = e^{3x}$.

4. Let $f(x) = e^{-x}$. First compute $T_3(x)$ and then use the error bound to show that $|f(x) - T_3(x)| \leq x^4/24$ for all $x \geq 0$.

5. Density and average value:
   (a) Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.
   (b) Find the average value of $f(x) = \sin(x)\cos(x)$ over $[0, \pi]$.

6. Volume of solid with known cross section:
   Calculate the volume of the following solid. The base is the region enclosed by $y = 2 - x^2$ and the x-axis. The cross sections perpendicular to the y-axis are squares.

7. Volumes:
   (a) (Disks) Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4. Use similar triangles to find the area of a horizontal cross section at a height y. Using this area, calculate the volume V by integrating the cross-sectional area.
   (b) (Washers) Let R be a region bounded by $y = x^2$ and $y = 1$, if R is rotated about x-axis, what is the volume of the resulting solid?
   (c) (Cylindrical Shells) V is obtained by rotating the region under the graph $y = 3x^2$ for $0 \leq x \leq 2$ about the y-axis. Calculate the volume of V.

8. Work:
   Calculate the work against gravity required to build a right circular cone of height 4 m and radius 2 m out of a lightweight material of density 600 kg/m^3. (See also question 7(a).)

9. Trigonometric Integrals:
   (a) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$
   (b) $\displaystyle\int \tan^3(x)\sec^3(x)\,dx$

MA 114 Worksheet # 17: Integration by trig substitution

1. Conceptual Understanding:

   (a) Given the identity $\sin^2\theta + \cos^2\theta = 1$, prove that:
   $$\sec^2\theta = \tan^2\theta + 1.$$

   (b) Given $x = a\sin(\theta)$ with $a > 0$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, show that $\sqrt{a^2 - x^2} = a\cos\theta$.

   (c) Given $x = a\tan(\theta)$ with $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, show that $\sqrt{a^2 + x^2} = a\sec\theta$.

   (d) Given $x = a\sec(\theta)$ with $a > 0$ and $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, show that $\sqrt{x^2 - a^2} = a\tan\theta$.

2. Compute the following integrals:

   (a) $\displaystyle\int_0^2 \frac{u^3}{\sqrt{16 - u^2}}\,du$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{25 - x^2}}\,dx$

   (c) $\displaystyle\int \frac{x^3}{\sqrt{64 + x^2}}\,dx$

   (d) $\displaystyle\int_0^1 \sqrt{x^2 + 1}\,dx$

   (e) $\displaystyle\int \frac{x}{\sqrt{x^2 + 1}}\,dx$

3. Let $a, b > 0$. Prove that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

4. Let $r > 0$. Consider the identity
   $$\int_0^s \sqrt{r^2 - x^2}\,dx = \frac{1}{2}r^2 \arcsin(s/r) + \frac{1}{2}s\sqrt{r^2 - s^2}$$
   where $0 \le s \le r$.

   (a) Plot the curves $y = \sqrt{r^2 - x^2}$, $x = s$, and $y = \dfrac{x}{s}\sqrt{r^2 - s^2}$.

   (b) Using part (a), verify the identity geometrically.

   (c) Verify the identity using trigonometric substitution.

MA 114 Worksheet # 18: Method of Partial Fractions & Numerical Integration

1. Write out the general form for the partial fraction decomposition but do not determine the numerical value of the coefficients.

   (a) $\dfrac{1}{x^2+3x+2}$

   (b) $\dfrac{x+1}{x^2+4x+4}$

   (c) $\dfrac{x}{(x^2+1)(x+1)(x+2)}$

   (d) $\dfrac{2x+5}{(x^2+1)^3(2x+1)}$

2. Compute the following integrals.

   (a) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

   (b) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

   (c) $\displaystyle\int \frac{x^3-2x^2+1}{x^3-2x^2}\,dx$

   (d) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

   (e) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

3. Compute

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$

   by first making the substitution $u=\sqrt[6]{x}$.

4. Conceptual Understanding:

   (a) Write down the Midpoint rule and illustrate how it works with a sketch.

   (b) Write down the Trapezoidal rule and the error bound associated with it.

5. Use the Midpoint rule to approximate the value of $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$ with $n=4$. Draw a sketch to determine if the approximation is an overestimate or an underestimate of the integral.

MA 114 Worksheet # 19: Numerical Integration

1. The velocity in meters per second for a particle traveling along the axis is given in the table below. Use the Midpoint rule to approximate the total distance the particle traveled from $t = 0$ to $t = 6$.

t	$v(t)$
0	0.75
1	1.34
2	1.5
3	1.9
4	2.5
5	3.2
6	3.0

2. Simpson's Rule turns out to exactly integrate polynomials of degree three or less. Show that Simpson's rule gives the *exact* value of $\int_0^h p(x)\,dx$ where $h > 0$ and $p(x) = ax^3 + bx^2 + cx + d$.

   [Hint: First compute the exact value of the integral by direct integration. Then apply Simpson's rule with n = 2 and observe that the approximation and the exact value are the same.]

3. Use the Midpoint Rule and then Simpson's Rule to approximate the integral $\int_0^{\pi} x^2 \sin x\,dx$ with $n = 8$. Compare your results to the actual value to determine the error in each approximation.

4. Use the Trapezoid Rule, Midpoint Rule and Simpson's Rule to approximate the integral $\int_0^2 \sqrt[4]{1 + x^2}\,dx$ with $n = 8$.

MA 114 Worksheet # 20: Arc Length and Surface Area

1. Conceptual Understanding:

   (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.

   (b) Write down the formula for the (surface) area of the surface obtained by rotating the graph of $f(x)$ about the x-axis for $a \leq x \leq b$. How would this formula change if the graph were instead rotated about $y = c$?

2. Find an integral expression for the arc length of the following curves. Do ***not*** evaluate the integrals.

   (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$.

   (b) $f(x) = x^4$ from $x = 2$ to $x = 6$.

   (c) $x^2 + y^2 = 1$

3. Find the arc length of the following curves.

   (a) $f(x) = x^{3/2}$ from $x = 0$ to $x = 2$.

   (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$.

4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.

5. Calculate the arc length of $f(x) = x^2$ over $[0, 1]$. [Hint: You will need to use a trigonometric substitution.]

6. Calculate the arc length of the graph of $f(x) = mx + r$ over $[a, b]$ in two ways: using the Pythagorean Theorem and using the arc length integral. [Hint: Make the arc of $f(x) = mx + r$ from $[a, b]$ the hypotenuse of a right triangle with legs $(b - a)$ and $m(b - a)$.]

7. Use Simpson's Rule with $n = 6$ to approximate the arc length of $f(x) = \sin(x)$ over $[0, \pi]$.

   For Problems 8–10, compute the surface area for a revolution about the x-axis over the given interval.

8. $y = x$, $[0, 4]$

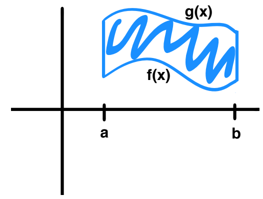
9. $y = x^3$, $[0, 2]$

10. $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0, 8]$

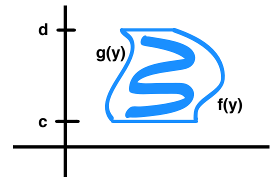
MA 114 Worksheet # 21: Center of Mass

1. Conceptual Understanding:

   (a) Write down the formulas for the coordinates of the centroid of a plate with constant density bounded between $x = a$, $x = b$, $f(x)$, and $g(x)$ as in the figure to the right.

   (b) Write down the formulas for the coordinates of the centroid of a plate with constant density bounded between $y = c$, $y = d$, $f(y)$, and $g(y)$ as in the figure to the right.

2. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$.

3. Point masses of equal size are placed at the vertices of the triangle with coordinates (3, 0), (b, 0), and (0, 6), where $b > 3$. Find the center of mass.

4. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \leq x \leq 1$. For practice, do this using both the approach from 1(a) and the approach from 1(b).

5. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.

6. Find the centroid of the region between $f(x) = x^{-1}$ and $g(x) = 2 - x$ for $1 \leq x \leq 2$.

7. Let $m > n \geq 0$. Find the centroid of the region between x^m and x^n for $0 \leq x \leq 1$. Find values for m and n that force the centroid to lie outside of the region.

MA 114 Worksheet # 22: Differential Equations and $y' = k(y - b)$

1. Conceptual Understanding:
   (a) What does it mean to say that a differential equation is first-order (or second-order or third-order. . .)
   (b) What does it mean to say that a differential equation is linear or nonlinear?

2. Use Separation of Variables to find the general solutions to the following differential equations.
   (a) $y' + 4xy^2 = 0$
   (b) $\sqrt{1 - x^2}y' = xy$
   (c) $(1 + x^2)y' = x^3y$
   (d) $\sqrt{1 + y^2}\, y' + \sec x = 0$

3. Solve $y' = 4y + 24$ subject to the condition that $y(0) = 5$.

4. Solve $y' + 6y = 12$ subject to the condition that $y(2) = 10$.

5. Recall that Newton's law of Cooling stipulates that the temperature $y(t)$ of a cooling object with respect to time satifies the differential equation

$$y' = -k(y - T_0),$$

   where k is a constant depending on the object and T_0 is the the temperature of the ambient environment.

   Frank's car engine runs at 210°F. On a 70°F day, he turns off the ignition and notes that five minutes later, the engine has cooled to 160°F.
   (a) Find the cooling constant k.
   (b) When will the engine cool to 100°F?

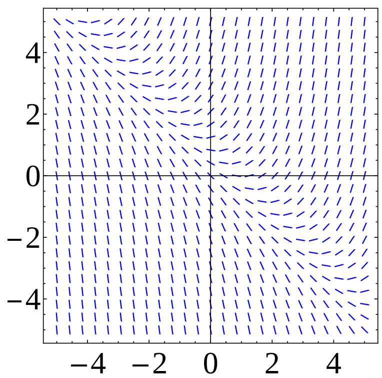
6. A cup of coffee with cooling constant $k = 0.09\text{min}^{-1}$ is placed in a room of temperature 20°C.
   (a) How quickly is the coffee cooling when the tempurature is 80°C?
   (b) Use the linear approximation to estimate the change in temperature over the next 6 s when the temperature is 80°C.
   (c) If the coffee is initially served at 90°C, how long will it take to reach an optimal drinking temperature of 65°C?

7. (Extra) A tank has the shape of the parabola $y = x^2$ revolved about the y-axis. Water leaks from a hole of area $B = 0.0005\ m^2$ at the bottom of the tank. Let $y(t)$ be the water level at time t. How long does it take for the tank to empty if the initial water level is $y(0) = 1\ m$?

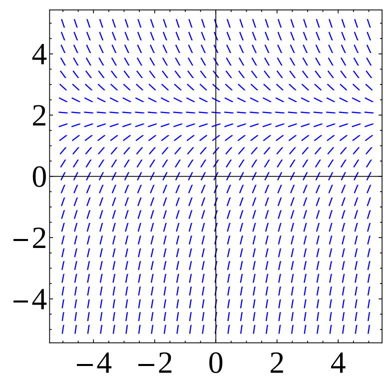
MA 114 Worksheet # 23: Graphical Methods

1. Match the differential equation with its slope field. Give reasons for your answer.

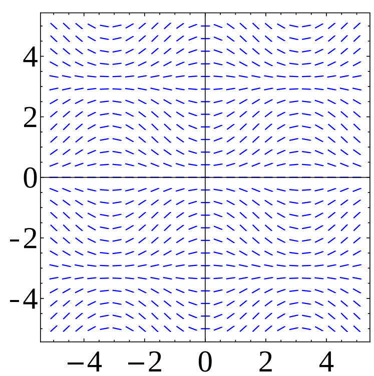
$$y' = 2 - y \qquad y' = x(2-y) \qquad y' = x + y - 1 \qquad y' = \sin(x)\sin(y)$$



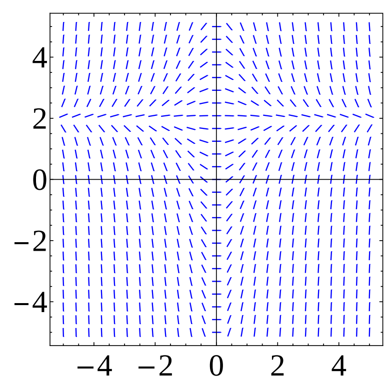
(a) Slope field I



(b) Slope field II



(c) Slope Field III



(d) Slope field IV

Figure 1: Slope fields for Problem 1

2. Use slope field labeled IV to sketch the graphs of the solutions that satisfy the given initial conditions

$$y(0) = -1, \quad y(0) = 0, \quad y(0) = 1.$$

3. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point

   (a) $y' = y - 2x$, $(1, 0)$

   (b) $y' = xy - x^2$, $(0, 1)$

4. Show that the isoclines of $y' = t$ are vertical lines. Sketch the slope field for $-2 \le t \le 2$, $-2 \le y \le 2$ and plot the integral curves passing through $(0, 1)$ and $(0, -1)$.

MA 114 Worksheet # 24: Review for Exam 3

1. Compute

   (a) $\int \frac{dx}{x^2 - 6x + 8}$

   (b) $\int \frac{3}{(x+1)(x^2+x)}\,dx$

   (c) $\int \frac{x^2}{x^2+9}\,dx$

   (d) $\int \frac{x^2+2}{x+3}\,dx$

2. Compute $\int \frac{e^x}{e^{2x} - e^x}\,dx$.
   Hint: First do a substitution, and then use partial fractions.

3. Evaluate $\int \frac{dx}{x^2 - 1}$ first with a trig substitution and then with partial fractions. Verify that the answer is the same in both cases.

4. Use trigonometric substitution to evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2 - 8}}$.

5. Recall the Trapezoid, Midpoint and Simpson's Rule.

   (a) Compute M_4 and T_4 for $\int_0^2 x^2\,dx$

   (b) Compute T_4 and S_4 for $\int_1^4 \frac{1}{x}\,dx$.

6. An airplane's velocity is recorded at 5 minute intervals during a 1 hour flight with the following results, in miles per hour: Estimate the total distance traveled by the plane during the hour using Simpson's Rule.
   $$\{550, 575, 600, 580, 610, 640, 625, 595, 590, 620, 640, 640, 630\}$$

7. Find the arc length of $f(x) = \ln(\sec(x))$ from $x = 0$ to $x = \pi/4$.

8. Find the surface area of the solid of revolution obtained by revolving $\sqrt{9 - x^2}$ about the x-axis for $-2 \le x \le 2$.

9. Consider point masses m_1, m_2, and m_3 centered at $(-1, 0)$, $(3, 0)$, and $(0, 4)$ respectively. If $m_1 = 6$, find m_2 so that the center of mass lies on the y-axis.

10. Use separation of variables to solve $y' + 4xy^2 = 0$.

11. Use separation of variables to solve $y' = (x+1)(y^2+1)$.

12. Find the solutions to $y' = -2y + 8$ subject to $y(0) = 3$ and $y(0) = 4$, respectively, and sketch their graphs.

13. Match each of the slope fields below with exactly one of the differential equations. (The scales on the x- and y-axes are the same.) Also, provide enough explanation to show why no other matches are possible.

(i) $y' = xy + 1$ (ii) $y' = xe^{-y}$ (iii) $y' = y^2 + 1$ (iv) $y' = \sin y$

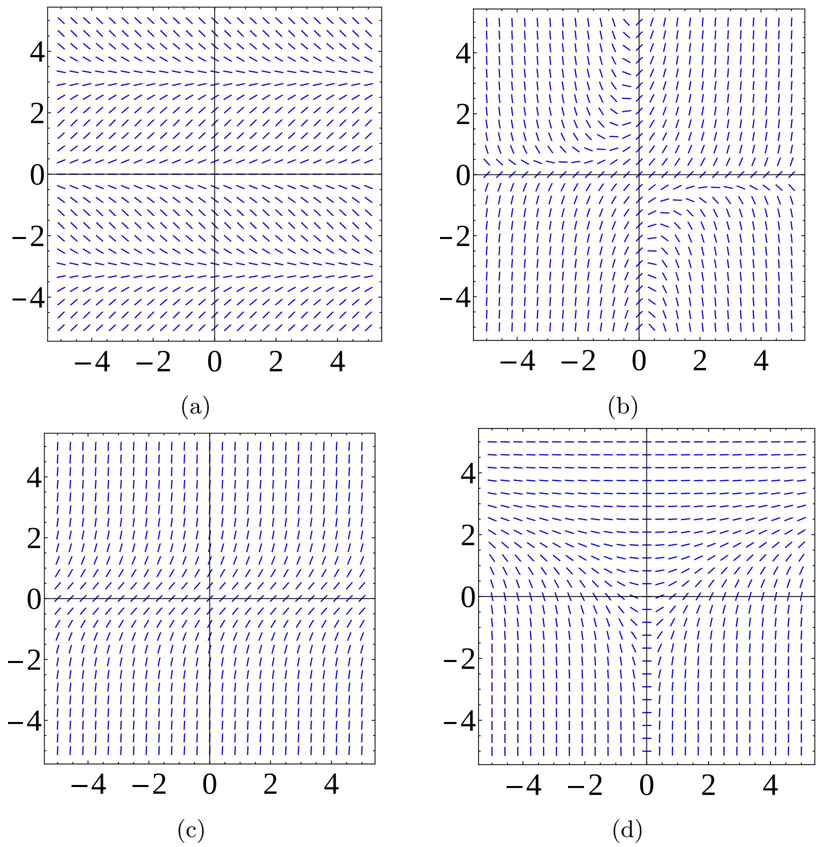


Figure 1: Slope fields for Problem 13

MA 114 Worksheet # 25:
The Logistic Equation and First-Order Linear Equations

1. The population of the world in 1990 was around 5.3 billion. Assume the growth constant is 1/265 and the carrying capacity is 100 billion.
   (a) Write out the logistic model and solve it.
   (b) Use this to estimate the population in 2014 and compare it with the actual population of 7.2 billion.
   (c) Use the logistic model to predict the population in 2100 and 2500.

2. Assume the carrying capacity of the U.S. population is 5 billion.
   (a) Use this and the fact that the population in 1990 was 250 million to find the logistic model for the U.S. population. (Do not solve for k).
   (b) Use the fact that the population in 2000 was 275 million to find k and $P(t)$.
   (c) Predict the U.S. population in 2100 and 2500
   (d) When will the U.S. population reach 350 million?

3. A lake with a carrying capacity of 10,000 fish is stocked with 400 fish. The number of fish triples in the first year.
   (a) Find the logistic model and solve it. (Also find k).
   (b) How long will it take for the population to reach 5000 fish?

4. Let $c > 0$. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

   where $k > 0$ is called a *doomsday equation* because $1 + c > 1$.
   (a) Use separation of variables to find the solution of this model with $y(0) = y_0$.
   (b) Show that there is a finite time $t = T$ (doomsday) such that

$$\lim_{t \to T^-} y(t) = \infty.$$

   (c) A certain breed of rabbits has the growth rate term $ky^{1.01}$. Suppose the initial population is 2 and there are 16 rabbits after 3 months. When is doomsday?

5. Consider $y' + x^{-1}y = x^3$.
   (a) Verify that $\alpha(x) = x$ is an integrating factor.
   (b) Show that when multiplied by $\alpha(x)$, the differential equation can be written as $(xy)' = x^4$.
   (c) Conclude that xy is an antiderivative of x^4 and use this information to find the general solution.
   (d) Find the particular solution satisfying $y(1) = 0$.

6. Solve the following differential equations:
   (a) $xy' = y - x$
   (b) $y' + 3x^{-1}y = x + x^{-1}$

7. Solve the following differential equations that satisfy the given initial condition:
   (a) $y' + 3y = e^{2x}$, $y(0) = -1$
   (b) $(\sin x)y' = (\cos x)y + 1$, $y(\pi/4) = 0$

MA 114 Worksheet # 26:
First-Order Linear Equations and Parametric Equations

1. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

2. Consider a series circuit consisting of a resistor of R ohms, an inductor of L henries and a variable voltage source of $V(t)$ volts (time t in seconds). The current through the circuit $I(t)$ (in amperes) satisfies the differential equation
$$\frac{dI}{dt} + \frac{R}{L} I = \frac{1}{L} V(t).$$
Assume that $R = 110\ \Omega$, $L = 10$ H, and $V(t) = e^{-t}$ volts.
   (a) Solve the equation with initial condition $I(0) = 0$,
   (b) Calculate t_m and $I(t_m)$, where t_m is the time at which $I(t)$ has a maximum value.

3. A tank with a capacity of 400 liters is full of a mixture of water and chlorine with a concentration of 0.05 grams of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 liters per second. The mixture is kept stirred and is pumped out at a rate of 10 liters per second. Find the amount of chlorine in the tank as a function of time.

4. Conceptual Understanding:
   (a) How is a curve different from a parametrization of the curve?
   (b) Suppose a curve is parametrized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

5. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \leq t \leq 2\pi$.
   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.
   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?
   (c) Use the above information to plot the curve.

6. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.
   (a) $x = \sqrt{t}$, $y = 1 - t$.
   (b) $x = 3t - 5$, $y = 2t + 1$.
   (c) $x = \cos(t)$, $y = \sin(t)$.

7. Represent each of the following curves as parametric equations traced just once on the indicated interval.
   (a) $y = x^3$ from $x = 0$ to $x = 2$.
   (b) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

8. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

MA 114 Worksheet # 27:
Tangent lines to parametric equations, arc length and speed

1. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.
   (b) $x = \cos(\theta) + \sin(2\theta)$, $y = \sin(\theta)$ at $\theta = \pi/2$.
2. For the following parametric curve, find dy/dx.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.
3. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \le t \le 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \le t \le 3$.
4. What is the speed of the parametrization $c(t) = (x(t), y(t))$? Use this to find the minimum speed of a particle with trajectory $c(t) = (t^2, 2\ln(t))$, for $t > 0$.
5. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius r centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define θ to be the angle formed by the x-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
   (a) Draw a picture and label θ.
   (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$.
   (c) Find the length of the involute for $0 \le \theta \le 2\pi$.
6. Consider the line through $P = (1, 0)$ and $Q = (7, 8)$. Find parametrizations of this line with the following speeds.
   (a) $s'(t) = 1$
   (b) $s'(t) = 3$
   (c) $s'(t) = t$
   (d) $s'(t) = t^2$

MA 114 Worksheet # 28:
Arc Length, Speed, Surface Area & Polar Coordinates

1. Consider the curve parametrized by $c(t) = (t^4, t^6)$
   (a) Find a cartesian equation for this curve.
   (b) Find the arc length for this curve for $0 \leq t \leq 1$. Which part of the curve given in part (a) does this compute?
   (c) Find the arc length for this curve for $-1 \leq t \leq 1$. Which part of the curve given in part (a) does this compute? How do you interpret your answer?

2. A "logarithmic spiral" is parametrized by $c(t) = (e^t \cos(t), e^t \sin(t))$.
   (a) Find the slope of the tangent lines, and use this to sketch this curve, for $0 \leq t \leq 2\pi$.
   (b) Find the speed $s'(t)$.
   (c) Find the length of the curve, again for $0 \leq t \leq 2\pi$.
   (d) What does the curve look like, for $-2\pi \leq t \leq 0$?

3. The curve parametrized by $c(t) = (\cos^3(t), \sin^3(t))$ is known as the "astroid".
   (a) Sketch this curve, for $0 \leq t \leq \pi$.
   (b) Find the length of this curve.
   (c) Find the area of the surface obtained by revolving the astroid around the x-axis.

4. Convert from rectangular to polar coordinates:
   (a) $(1, \sqrt{3})$
   (b) $(-1, 0)$
   (c) $(2, -2)$

5. Convert from polar to rectangular coordinates:
   (a) $\left(2, \frac{\pi}{6}\right)$
   (b) $\left(-1, \frac{\pi}{2}\right)$
   (c) $\left(1, -\frac{\pi}{4}\right)$

6. Sketch the graph of the polar curves:
   (a) $\theta = \frac{3\pi}{4}$
   (b) $r = \pi$

7. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{3}$.

8. Find the polar equation for:
   (a) $x^2 + y^2 = 9$
   (b) $x = 4$
   (c) $y = 4$

9. Convert the equation of the circle $r = 2\sin\theta$ to rectangular coordinates and find the center and radius of the circle.

10. Given the circle represented by $x^2 + (y-2)^2 = 4$
    (a) Find the polar representation for this equation.
    (b) Calculate the area enclosed by $0 \leq \theta \leq \pi/4$.
    (c) Sketch the area calculated.

MA 114 Worksheet # 29:
Final Exam Review

Caution: This review sheet does not cover all the possible problems you may see on the final exam. Be sure to review all topics listed on the course calendar.

1. Integration: Compute each of the following (unless the integral is divergent).
   (a) $\displaystyle\int \frac{\sin(\ln(t))}{t}\,dt$
   (b) $\displaystyle\int e^x \sin x\,dx$
   (c) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$
   (d) $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$
   (e) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$
   (f) $\displaystyle\int_{-\infty}^0 xe^x\,dx$

2. How many terms should you take in Simpson's rule to approximate $\displaystyle\int_1^2 (\sin x + x + 1)\,dx$ correct to 5 decimal places?

3. Areas, Volumes, and Lengths: Set up (but do not evaluate) integrals for the following geometric quantities.
   (a) The area enclosed by the curves $y = 1 - 2x^2$, $y = |x|$.
   (b) The volume obtained by rotating the region bounded by the curves $y = 1/x$, $x = 1$, and $x = 5$ about the x-axis.
   (c) The volume obtained by rotating the region bounded by the curves $y = 1/x$, $x = 1$, and $x = 5$ about the y-axis.
   (d) The volume obtained by rotating the region bounded by the curves $y = \tan x$, $y = x$, and $x = \pi/3$ about the y-axis.
   (e) The length of the curve $y = x^2$ from $x = a$ to $x = b$.
   (f) The length of the parametric curve $x = 3t^2$, $y = 2t^3$, $0 \le t \le 2$.
   (g) The area enclosed by the polar curve $r = 1 - \cos\theta$.

4. Parametric equations:
   (a) Consider the curve $x(t) = 3t^2 + t$ and $y(t) = 2t$
      i. Eliminate the parameter, t, to find a Cartesian equation for the curve.
      ii. Find the tangent line to this curve at the point $(x, y) = (14, 4)$.
   (b) Consider the polar curves $r = \sin(2\theta)$ and $r = \cos\theta$.
      i. Determine the Cartesian coordinates (x, y) of the point of intersection which is strictly in the first quadrant, i.e. $x, y > 0$.
      ii. Set up an integral, or integrals, for computing the area of the region in the first quadrant between the bolded portion of the two curves. Do not evaluate the integral(s).

5. Differential Equations

   (a) Solve $\dfrac{dL}{dt} = kL^2 \ln t,\ L(1) = -1$.

   (b) Draw the direction field for the differential equation $y' = y + x$. Sketch the solution which satisfies $y(0) = 0$.

   (c) Find the solution of $y' - y = e^{2x},\ y(0) = 1$.

   (d) Consider the differential equation $y' = x^2 - y$. If at the n-th iteration of Euler's method with $h = 0.1$ we have $(x_n, y_n) = (3.1, -2)$, what is (x_{n+1}, y_{n+1})?

6. Sequences and Series

   (a) Does the sequence $\left\{ \dfrac{2+n^3}{4+5n^3} \right\}$ converge? If so, what is its limit?

   (b) True or false: If $\lim_{n\to 1} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges?

   (c) Does the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^{1/3}}$ converge conditionally, converge absolutely, or diverge? Explain.

   (d) $\sum_{n=0}^{\infty} \left(\dfrac{1}{n+1} - \dfrac{1}{n+2} \right)$.

   (e) Test the following series for convergence.

      i. $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$

      ii. $\sum_{n=1}^{\infty} \dfrac{n^7}{7^n}$

      iii. $\sum_{n=0}^{\infty} \dfrac{\cos(n)}{2+2^n}$

      iv. $\sum_{n=1}^{\infty} \dfrac{5^n + n^2 + n + 17}{3n^4 + 4^n + 1 + 5}$

7. Power, Maclaurin, and Taylor Series

   (a) Find the Maclaurin series for $\dfrac{x^2}{1+x}$.

   (b) Find the Taylor series for $\cos x$ about $a = \pi/2$.

8. Use the limit comparison test to determine if $\sum_{n=1}^{\infty} \left(1 - \cos \dfrac{1}{n}\right)$ converges or diverges.(Hint: compare with $\sum \dfrac{1}{n^2}$.