MA 113 — Calculus I Fall 2009
Exam 2 October 20, 2009

Answer all of the questions 1 - 7 and two of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Answer key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		10
2		8
3		13
4		10
5		10
6		8
7		8
8		15
9		15
10		15
Free	3	3
		100

(1) Find the derivative of the following functions.

(a) $g(x) = \dfrac{x^3}{x^2+1}$.

(b) $h(t) = t^2 \cdot e^{t^2+1}$.

(c) $f(x) = \tan(3x)$

3

(a) $g'(x) = \dfrac{(x^2+1)3x^2 - x^3 \cdot 2x}{(x^2+1)^2}$

$= \dfrac{x^4+3x^2}{(x^2+1)^2}$

4

(b) $h'(t) = 2 \cdot t \cdot e^{t^2+1} + t^2 \cdot (2t) e^{t^2+1}$

$= 2t(1+t^2) e^{t^2+1}$

3

(c) $f'(x) = \sec^2(3x) \cdot 3$

(a) $g'(x) =$ $\dfrac{x^4+3x^2}{(x^2+1)^2}$

(b) $h'(t) =$ $2t \cdot (1+t^2) e^{t^2+1}$

(c) $f'(x) =$ $\sec^2(3x) \cdot 3$

(2) Given that
$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$
use the limit laws to find the limits.

(4)

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}$. $\displaystyle = \lim_{x\to 0}\left(x\cdot\frac{\sin(x^2)}{x^2}\right)$ (1)

$\displaystyle = \lim_{x\to 0} x \cdot \lim_{x\to 0}\frac{\sin(x^2)}{x^2} \underset{t=x^2}{=} \lim_{x\to 0} x \cdot \lim_{t\to 0}\frac{\sin(t)}{t}$ (2)

$= 0\cdot 1 = \underline{\underline{0}}$ (1)

(4)

(b) $\displaystyle\lim_{x\to 0}\left(e^{x^3+1}\frac{\sin(4x)}{x}\right) \underset{(1)}{=} \lim_{x\to 0}\left(4\,e^{x^3+1}\cdot\frac{\sin(4x)}{4x}\right)$

$\displaystyle = 4\cdot\underbrace{\lim_{x\to 0} e^{x^3+1}}_{=e^{0}=e}\cdot\underbrace{\lim_{x\to 0}\frac{\sin(4x)}{4x}}_{1}$ (2)

$= \underline{\underline{4e}}$ (1)

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x} =$ ___0___

(b) $\displaystyle\lim_{x\to 0}\left(e^{x^3+1}\frac{\sin(4x)}{x}\right) =$ ___4e___

(3) A particle is traveling along a straight line. Its position after $t \geq 0$ seconds is given by

$$s(t) = \frac{1}{3}t^3 - t^2 - 3t + 1.$$

meters. **As usual, justify your answer.**

(a) Find the time interval(s) where the particle is traveling in the positive direction?

2 [$v(t) = s'(t) = t^2 - 2t - 3 = (t-3)(t+1)$

For $t \geq 0$ the factor $t+1$ is always > 0.

2 [Therefore, $v(t) > 0$ if and only if $t - 3 > 0$ if and only if $t > 3$.

$(3, \infty)$

(b) What is the total distance traveled by the particle during the first 6 seconds?

2 [backward between 0 and 3 and forward from 3 to 6. Therefore,

2 [total distance $= |s(3) - s(0)| + |s(6) - s(3)|$

1 [$= |9 - 9 - 9 + 1 - 1| + |72 - 36 - 18 + 1 - 9 + 9 + 9 - 1|$

$= 9 + 27 = 36$

(c) Find the time interval(s) where the particle is speeding up?

2 [$a(t) = 2t - 2 = 2(t-1)$; $a(1) = 0$

2

	0 to 1	1 to 3	3 and beyond
$a(t)$	−	+	+
$v(t)$	−	−	+

speeding up on $(0, 1)$ and $(3, \infty)$

(a) Time interval(s) $(3, \infty)$

(b) Total distance 36 meters

(c) Time interval(s) $(0, 1)$ and $(3, \infty)$

(4) Use the differentiation rules to determine the following higher order derivatives. **As always, show your work.**

(a) Find $f''(x)$ if $f(x) = \ln(3x^2 + 5x - 4)$.

(2) $f'(x) = \dfrac{6x+5}{3x^2+5x-4}$

(2) $f''(x) = \dfrac{(3x^2+5x-4)\cdot 6 - (6x+5)(6x}{(3x^2+5x-4)^2}$

not required $= \dfrac{-18x^2-30x-49}{(3x^2+5x-4)^2}$

(b) Find $g''(x)$ if $g(x) = (2x+1)\cdot\sin(3x)$

(3) $g'(x) = 2\cdot\sin(3x) + (2x+1)\cdot 3\cos(3x)$
$= 2\cdot\sin(3x) + (6x+3)\cos(3x)$

(3) $g''(x) = 2\cdot 3\cos(3x) + 6\cos(3x)$
$+ (6x+3)\cdot 3\cdot(-\sin(3x))$

not required $= 12\cos(3x) - (18x+9)\sin(3x)$

(a) $f''(x) =$ $\dfrac{-18x^2-30x-49}{(3x^2+5x-4)^2}$

(b) $g''(x) =$ $12\cos(3x) - (18x+9)\sin(3x)$

(5) Consider the curve described by the equation $xy^2 + x^2y + y^3 = 7$. Find the equation of the line tangent to this curve at the point $(2, 1)$. Write your answer in the form $y = mx + b$. **As always, show your work.**

Implicit differentiation:

3 $$y^2 + x \cdot 2 \cdot y \cdot y' + 2xy + x^2 y' + 3y^2 y' = 0$$

2 $$y' = \frac{-y^2 - 2xy}{2xy + x^2 + 3y^2}$$

Slope at $(2,1)$ is

2 $$y'(2) = \frac{-1-4}{4+4+3} = \frac{-5}{11}$$

Equation of tangent line

2 $$y - 1 = -\frac{5}{11}(x-2)$$

1 $$y = -\frac{5}{11}x + \frac{21}{11}$$

Equation of the tangent line is $y = -\frac{5}{11}x + \frac{21}{11}$

(6) As usual show your work in answering the following questions. **Give the exact answer!**

(a) Find $F'(\pi/4)$ where $F(x) = \ln|\sin(x)|$.

2 $F'(x) = \dfrac{\cos x}{\sin x}$

2 $F'\left(\frac{\pi}{4}\right) = \dfrac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

(b) Find $G'(2)$ where $G(x) = \arctan(e^x)$

$\arctan'(x) = \dfrac{1}{1+x^2}$

2 $G'(x) = \dfrac{1}{1+(e^x)^2} \cdot e^x$

$= \dfrac{e^x}{1+e^{2x}}$

2 $G'(2) = \dfrac{e^2}{1+e^4}$

(a) 1

(b) $\dfrac{e^2}{1+e^4}$

(7) A certain bacteria culture is known to grow exponentially, that is, its population at time t is given by a function of the form

$$p(t) = p_0 e^{kt},$$

where k is a constant and p_0 is the initial population. It has been found that the culture tripled in size in 10 hours. When did it double in size? **Give the exact answer.**

(2) $p(10) = p_0 e^{10k} = 3p_0$ (tripling)

(2)
$$e^{10k} = 3$$
$$10k = \ln(3)$$
$$k = \frac{\ln(3)}{10}$$

Thus, the population function is

$$p(t) = p_0 e^{\frac{\ln(3)}{10} \cdot t}.$$

For doubling:

(2) $$p(t) = p_0 e^{\frac{\ln(3)}{10} \cdot t} = 2p_0$$

(2)
$$e^{\frac{\ln(3)}{10} \cdot t} = 2$$
$$\frac{\ln(3)}{10} \cdot t = \ln(2)$$
$$t = \frac{\ln(2) \cdot 10}{\ln(3)}$$

Not required: $\frac{\ln(2) \cdot 10}{\ln(3)} \approx 6.309$, hence the population doubled after ≈ 6.3 hours.

(a) It doubled in $\frac{\ln(2) \cdot 10}{\ln(3)}$ hours.

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) Consider the function $f(x) = 3x^2 + 6$.

(a) Find the equation of the tangent line to the graph of $f(x)$ at the point $(a,\ f(a))$.

2 $f'(x) = 6x$, $f'(a) = 6a$

Point - slope formula for point $(a, f(a))$ and slope $6a$:

4 $y - f(a) = 6a(x-a)$

$y = 6ax - 6a^2 + \underbrace{f(a)}_{3a^2+6}$

2 $\boxed{y = 6ax - 3a^2 + 6}$

(b) Find all points $(a, f(a))$ on the graph of $f(x)$ such that the tangent line to the graph at $x = a$ passes through the point $(2, -9)$. **As usual, show your work to support your answer.**

$(2,-9)$ has to lie on the line found in (a). Thus,

3 $-9 = 6a \cdot 2 - 3a^2 + 6$

2 $3a^2 - 12a - 15 = 0$

$a^2 - 4a - 5 = 0$

$(a-5)(a+1) = 0$, $a = 5$, $a = -1$.

Hence we find the points

2 $(a, f(a)) = (5, f(5)) = (5, 81)$

$(a, f(a)) = (-1, f(-1)) = (-1, 9)$

(a) Equation of the tangent line: $y = 6ax - 3a^2 + 6$

(b) The points are: $(5, 81)$, $(-1, 9)$

(9) (a) State the Chain Rule. Use complete sentences and include all assumptions necessary to make the rule valid.

1

5

If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g = F$ is differentiable at x and

1 1

$$F'(x) = f'(g(x)) \cdot g'(x).$$

2

(b) Suppose f and g are differentiable functions such that

$$f(6) = 2, \; f'(6) = 4, \; g(2) = 3, \text{ and } g'(2) = -7.$$

Find each of the following.

(a) $h'(6)$ where $h(x) = \arctan\left(f(x)^2\right)$.

$$\boxed{\arctan'(x) = \frac{1}{1+x^2}}$$

3

$$h'(x) = \frac{1}{1+f(x)^4} \cdot 2f(x) \cdot f'(x)$$

2

$$h'(6) = \frac{1}{1+2^4} \cdot 2 \cdot 2 \cdot 4 = \underline{\underline{\frac{16}{17}}}$$

(b) $k'(2)$ where $k(x) = f\left(\frac{x^3}{4} \cdot g(x)\right)$

3

$$k'(x) = f'\left(\frac{x^3}{4} \cdot g(x)\right) \cdot \left[\frac{3x^2}{4} g(x) + \frac{x^3}{4} g'(x)\right]$$

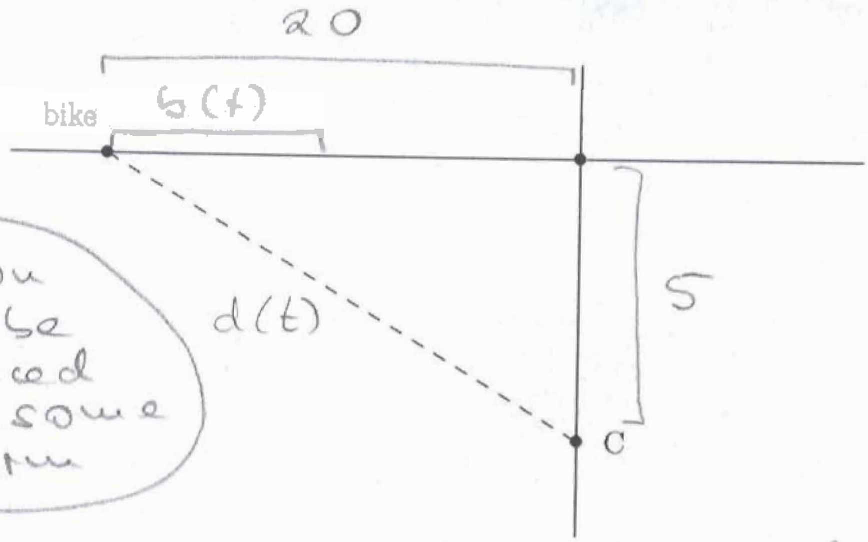
2

$$k'(2) = f'\left(\frac{8}{4} \cdot 3\right) \cdot \left[\frac{3 \cdot 4}{4} \cdot 3 + \frac{8}{4}(-7)\right]$$

$$= f'(6) \cdot (9 - 14)$$

$$= 4 \cdot (-5) = \underline{\underline{-20}}$$

(10) A cyclist starts 20 km west of an intersection and rides with a constant speed of 40 kilometers per hour toward the intersection. A cappuccino bar is located 5 kilometers south of the intersection at point C. At what rate is the distance between the cyclist and the cappuccino bar decreasing when the cyclist is halfway to the intersection?



Notation has to be introduced in some form

(2) $b(t)$ = distance traveled by the cyclist at time t

$d(t)$ = distance between cyclist and bar at time t.

Known: $b'(t) = 40$ for

Wanted: $d'(t_0)$ where t_0 is

$b(t_0) = 10$

(3) Relation: $d^2(t) = (20 - b(t))^2 + 25$

(2) Differentiate: $2d(t)\, d'(t) = 2(20 - b(t)) \cdot b'(t)$

(3) $d(t)\, d'(t) = -40 \cdot (20 - b(t))$

$d'(t) = \dfrac{-40 \cdot (20 - b(t))}{d(t)}$

(3) At time t_0: $b(t_0) = 10$,

$d(t_0) = \sqrt{(20-10)^2 + 25} = \sqrt{125} = 5\sqrt{5}$. Hence

(a) Distance decreases at a rate of: $\frac{80}{\sqrt{5}}$ km/h

(2) $d'(t_0) = \dfrac{-40 \cdot 10}{5\sqrt{5}} = \dfrac{-80}{\sqrt{5}}$