MA 113 — Calculus I
Exam 3

Fall 2009
November 17, 2009

Answer all of the questions 1 - 7 and two of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: KEY

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		9
2		8
3		13
4		8
5		10
6		8
7		10
8		15
9		15
10		15
Free	4	4
		100

(1) (9 points) Consider the function

$$g(x) = \frac{x}{x^2+1}$$

which has derivatives

$$g'(x) = \frac{(1-x^2)}{(x^2+1)^2}$$

and

$$g''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Give exact answers to the following questions!

(a) (6 points) Find the intervals where the graph of g is concave up and concave down.

① $g''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = \frac{2x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)3}$

① Sign graph: $\overset{-}{}\ \underset{-\sqrt{3}}{+}\ \overset{+}{}\ \underset{0}{+}\ \overset{-}{}\ \underset{\sqrt{3}}{+}\ \overset{+}{}$

x	$2x(x+\sqrt{3})(x-\sqrt{3})$	
-2	$-4(-2+\sqrt{3})(-2-\sqrt{3})$	<0
-1	$-2(-1+\sqrt{3})(-1-\sqrt{3})$	>0
$+1$		<0
$+2$		>0

② Concave up: $g''(x) > 0$ $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$

② Concave down: $g''(x) < 0$ $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

(b) (3 points) Find the inflection points of g (x- and y-coordinates).

Inflection pts occur when g'' changes sign.

From the sign graph of g'' above, then

@① occur at $x = -\sqrt{3}$, $x = 0$, $x = \sqrt{3}$

The y-values are $g(-\sqrt{3}) = -\frac{\sqrt{3}}{4}, 0, \frac{\sqrt{3}}{4}$

(a) Interval(s) where the graph of g is

concave up: $(-\sqrt{3}, 0)$ $(\sqrt{3}, \infty)$

concave down: $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

(b) Inflection point(s): $-\sqrt{3}$, 0, $\sqrt{3}$ y-values $-\frac{\sqrt{3}}{4}, 0, \frac{\sqrt{3}}{4}$

(2) (8 points) Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 6x^2 + 9x + 2$$

on the interval $[-1, 4]$.

f is everywhere differentiable, so critical pts occur where $f'(x) = 0$

② $f'(x) = 3x^2 - 12x + 9$ ∴ critical points at

$= 3(x^2 - 4x + 3)$ $x = 1$, $x = 3$

$= 3(x-1)(x-3)$

② Test f at critical points and end points: (closed interval test)

①

x	f(x)
-1	-14
1	6
3	2
4	6

① The absolute minimum is $f(-1) = -14$

② '' '' maximum is $f(1) = 6 = f(4)$

Both values of x for full credit

5

(a) Absolute maximum value 6 at $x =$ 1, 4

(b) Absolute minimum value −14 at $x =$ −1

(3) (13 points) Consider the function $f(x) = \ln(x^2 - 6x + 11)$.
Give exact answers to the following questions!

(a) (4 points) Find the domain of f.

Domain is $\{x : x^2 - 6x + 11 > 0\}$

Since $x^2 - 6x + 11 = x^2 - 6x + 9 + 2 = (x-3)^2 + 2$,

$x^2 - 6x + 11 \geq 2$ so the domain of f is all $x \in \mathbb{R}$

(b) (3 points) Find the critical number(s) of f.

$f'(x) = \dfrac{2x-6}{x^2-6x+11}$ $\qquad$ $f'(x) = 0$ if $2x - 6 = 0$

$\therefore$ $2x = 6$

$\therefore$ $x = 3$

(c) (3 points) Find the intervals of increase and decrease for f.

The sign of $f'(x)$ is the sign of $2x-6$, so:

f' $\quad - \quad 3 \quad +$

One can also check:

x	$f'(x)$	
0	-6	$(-)$
4	$2/3$	$(+)$

Increase: $(3, \infty)$

Decrease $(-\infty, 3)$

(d) (3 points) Find the local extremum or extrema of f. Be sure to give both the x-coordinate and the value of $f(x)$. Indicate whether each is a local maximum or a local minimum.

The only local extremum occurs at $x = 3$.

Here $f(3) = \ln(3^2 - 6 \cdot 3 + 11) = \ln(9 - 18 + 11) = \ln(2)$

By the first derivative test, $x = 3$ is a local minimum

(a) Domain: $(-\infty, \infty)$

(b) Critical number(s): $x = 3$

(c) Interval(s) of increase: $(3, \infty)$ and decrease: $(-\infty, 3)$

(d) Local extremum/extrema: $x = 3$, $f(3) = \ln 2$, local minimum

(4) (8 points) Let $g(x) = \sqrt{25+x}$. Find the linear approximation to g at $x = 0$. Use the linear approximation to estimate $\sqrt{25.06}$.

① $g(0) = 5$

The linear approximation

$L(x) = g(0) + g'(0) \cdot x$

② $g'(x) = \frac{1}{2\sqrt{25+x}}$

$\therefore g'(0) = \frac{1}{2 \cdot 5} = \frac{1}{10}$

② $\therefore L(x) = 5 + \frac{x}{10}$

③ $\sqrt{25.06} = g(0.06)$ exactly.

By linear approximation, $g(0.06) \approx L(0.06)$

$= 5 + \frac{0.06}{10}$

$= 5.006$

Linear approximation $L(x) = 5 + \frac{x}{10}$, $\sqrt{25.06} \approx$ 5.006

[redacted]

(5) (10 points) Find the general antiderivative of each of the following functions.

(a) (3 points) $f(t) = 3\sec^2(t) + \cos(t)$

③ $F(t) = 3\tan(t) + \sin(t) + C$

(b) (3 points) $g(x) = x^2 - 2x + 4e^x$

③ $G(x) = \frac{x^3}{3} - x^2 + 4e^x + C$

(c) (4 points) $h(w) = w^{3/2} + 1/w$

④ $H(w) = \frac{2}{5}w^{5/2} + \ln|w| + C$

(a) General antiderivative of $f(t)$ is $3\tan(t) + \sin(t) + C$

(b) General antiderivative of $g(x)$ is $\frac{x^3}{3} - x^2 + 4e^x + C$

(c) General antiderivative of $h(w)$ is $\frac{2}{5}w^{5/2} + \ln|w| + C$

(6) (8 points) A particle is moving with acceleration $a(t) = 10\sin(t) + 3\cos(t)$ meters/sec^2. Its initial position is $s(0) = 0$ meters and its initial velocity is $v(0) = 5$ meters per second.

(a) (4 points) Find $v(t)$, the velocity of the particle, at any time t. Be sure to specify the units.

$$v(t) = \int [10\sin(t) + 3\cos(t)]\,dt$$

② $$= -10\cos(t) + 3\sin(t) + C_1$$

Since $v(0) = 5$ m/sec,

$$-10\cdot 1 + 3\cdot 0 + C_1 = 5$$

② $$C_1 = 15$$

$$\therefore v(t) = -10\cos(t) + 3\sin(t) + 15 \quad \text{m/sec}$$

(b) (4 points) Find $s(t)$, the position of the particle, at any time t. Be sure to specify the units.

$$s(t) = \int (-10\cos(t) + 3\sin(t) + 15)\,dt$$

② $$= -10\sin(t) - 3\cos(t) + 15t + C_2$$

$s(0) = 0$ so

$$-10\cdot 0 - 3\cdot 1 + 15\cdot 0 + C_2 = 0$$

② $$\therefore C_2 = 3$$

$$s(t) = -10\sin(t) - 3\cos(t) + 15t + 3 \quad \text{m}$$

No units (-1)

(a) $v(t) =$ $-10\cos(t) + 3\sin(t) + 15$ m/sec

(b) $s(t) =$ $-10\sin(t) - 3\cos(t) + 15t + 3$ m

(7) (10 points) Let $g(x) = \dfrac{3x^2+1}{x^2-x-6}$

(a) (4 points) Find the horizontal asymptotes, if any. Be sure to justify your work!

$$\lim_{x\to+\infty} \frac{3x^2+1}{x^2-x-6} = \lim_{x\to+\infty} \frac{6x}{2x-1} = \lim_{x\to+\infty} \frac{6}{2} = \boxed{3}$$

(+3) ✓

by L'Hospital, (type ∞/∞) or $\lim_{x\to\infty} \frac{3x^2+1}{x^2-x-6} = \lim_{x\to\infty} \frac{x^2(3+1/x^2)}{x^2(1-1/x-6/x^2)} = 3$

Similarly

$$\lim_{x\to-\infty} \frac{3x^2+1}{x^2-x-6} = \lim_{x\to-\infty} \frac{6x}{2x-1} = \lim_{x\to-\infty} \frac{6}{2} = \boxed{3}$$

(+1) So $y=3$ is the unique horizontal asymptote

(b) (6 points) Find all vertical asymptotes. Find the left- and right-hand limits of g at each vertical asymptote. Be sure to justify your work!

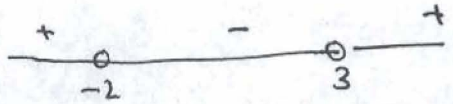
(+1) $g(x) = \dfrac{3x^2+1}{(x+2)(x-3)}$ so vertical asymptotes occur at

$x=-2$ and $x=3$

Since $3x^2+1>0$ the left-and right-hand limits at $x=-2$ and $x=3$

(+1) are determined by the sign of the denominator $(x+2)(x-3)$:

Sign $(x+2)(x-3)$:



Limits @ (+1)

Hence $\lim_{x\to-2^-} g(x) = +\infty$ $\qquad \lim_{x\to3^-} g(x) = -\infty$

$\lim_{x\to-2^+} g(x) = -\infty$ $\qquad \lim_{x\to3^+} g(x) = +\infty$

(a) Horizontal asymptote(s) $y=3$

(b) Vertical asymptote(s) $x=-2,\ x=+3$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (15 points)

(a) (5 points) State the Mean Value Theorem. Be sure to state all hypotheses as well as the conclusion.

① ①

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then, there exists a ① point $c \in (a,b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad \text{formula ②}$$

(b) (5 points) Professor Perry drove from Lexington to Columbus, Ohio (a distance of 200 miles) on interstate highways where the speed limit is 65 miles per hour. On arrival in Columbus, he proudly told a former Calculus I student who is now a police officer that he had made the drive in only 2.5 hours. The officer promptly gave him a speeding ticket. Using the Mean Value Theorem, explain how the police officer knew that Perry had been speeding.

① Let $s(t)$ be Perry's distance from Lexington.

From the data:

① $s(0) = 0$ miles

$s(2.5) = 200$ miles

Applying the M.V.T. to $s(t)$ on $[0, 2.5]$, we see that for some time c between 0 and 2.5,

② $$s'(c) = v(c) = \frac{200-0}{2.5-0} = 80 \ \frac{\text{miles}}{\text{hour}}$$

That is, at some time during the trip, Perry

① was driving at 80 miles per hour.

(c) (5 points) Suppose that f is a function continuous on $[0, 3]$, differentiable on $(0, 3)$, that $-1 \leq f'(x) \leq 4$, and $f(0) = 5$. Based on this information, is it possible that $f(3) = 20$? Explain why or why not using the Mean Value Theorem.

(+1) Apply the MVT to f on $[0,3]$. (since f is continuous on $[0,3]$, differentiable on $(0,3)$).

- - - - -

(+1) $$\frac{f(3)-f(0)}{3-0} = f'(c) \text{ for some } c \in (0,3)$$

(+3)

so $$-1 \leq \frac{f(3)-f(0)}{3-0} \leq +4$$

$$\therefore -3 \leq f(3) - f(0) \leq 12$$

$$\therefore -3 \leq f(3) - 5 \leq 12$$

$$\therefore 2 \leq f(3) \leq 17$$

Hence, it is not possible that $f(3) = 20$.

- - - - - - - -

Alternatively, one can argue by contradiction: if $f(3)=20$,

(+3) $$\frac{f(3)-f(0)}{3-0} = \frac{20-5}{3} = \frac{15}{3} = f'(c) \text{ for some c in } (0,3)$$

(+1) Since $\frac{15}{3} > 4$, it cannot be the case that $f(3) = 20$.

(9) (15 points)

(a) (5 points) State L'Hospital's Rule.

① Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly a).

② Suppose that either (1) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or (2) $\lim_{x\to a} f(x) = \pm\infty$ ~~as~~ and $\lim_{x\to a} g(x) = \pm\infty$. Then

② $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the latter limit exists or is ±∞.

(b) (5 points) Find a and b so that

$$\lim_{x\to 0} \frac{e^{4x} - a - bx}{x^2}$$

exists (that is, is finite), and find the value of the limit for these numbers a and b.

(1) Let $f(x) = e^{4x} - a - bx$, $g(x) = x^2$

To have an indeterminate form of type $0/0$, we need $f(0) = 1 - a = 0$ so $\boxed{a = 1}$

(+2)

(2) By L'Hospital's rule,

$\lim_{x\to a}\left(\frac{e^{4x}-a-bx}{x^2}\right) = \lim_{x\to a} \frac{4e^{4x}-b}{2x}$ if the latter limit exists. To have an indeterminate form of type $0/0$, we must have $4 - b = 0$ or $\boxed{b = 4}$

(+2)

(3) By the above steps, $\lim_{x\to 0}\left(\frac{e^{4x}-1-4x}{x^2}\right) = \lim_{x\to 0}\left(\frac{4e^{4x}-4}{2x}\right)$

$= \lim_{x\to 0}\left(\frac{16e^{4x}}{2}\right) = 8$

(+1)

$a =$ 1, $b =$ 4, limit = 8

(c) (5 points) Use L'Hospital's rule to determine

$$\lim_{x \to 0} \frac{\sin(3x)}{\tan(6x)}.$$

$f(x) = \sin(3x)$

(+1) $g(x) = \tan(6x)$

(+1) Note $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$, so limit is an indeterminate form of type $\frac{0}{0}$.

(+3)

$$\therefore \lim_{x \to 0} \frac{\sin(3x)}{\tan(6x)} = \lim_{x \to 0} \frac{3\cos(3x)}{6\sec^2(6x)}$$

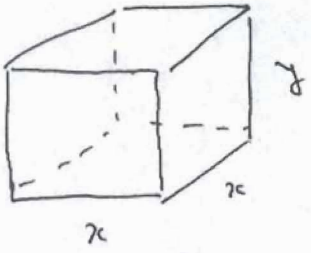
$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

where we've used the substitution principle to evaluate the last limit.

The limit is 1/2

(10) (15 points) A box with a square base and an open top is to have a volume of 180cm^3. Material for the base costs \$5 per cm^2 and material for the sides costs \$3 per cm^2. Find the dimensions of the cheapest box.



② Let x = width of box in cm

y = height of box in cm

(+2) The volume constraint implies that

$$x^2 y = 180$$

(+2) Area of base $= x^2$

Area of 4 sides $= 4xy$

cost $= Q = 5x^2 + 3 \cdot 4xy$ dollars

(+2) using the constraint (*), $y = \frac{180}{x^2}$

$$\therefore Q(x) = 5x^2 + 12x \cdot \left(\frac{180}{x^2}\right)$$

$$= 5x^2 + \frac{2160}{x}$$

(+1) Domain: $x \in (0, \infty)$

(+3) $Q'(x) = 10x - \frac{2160}{x^2} = \frac{10x^3 - 2160}{x^2}$

$Q'(x) = 0$ if $x^3 = 216$ or $x = 6$

Sign graph of $Q'(x)$ — 6 +

x	$Q'(x)$
1	$10 - 2160 <$
10	$100 - \frac{2160}{100} >$

(+1) By first derivative test, $x = 6$ is an absolute min

Note if $x = 6$, $36 \cdot y = 180$ or $y = 5$

Dimensions are $x = 6$ cm, $y = 5$ cm

Answers (+1) each

No units (−1)