Solution to Assignment 2 for MA 113 - Calculus I

(1) (3 Points) Let x be any real number except -7 and let $f(x) = \dfrac{6}{x+7}$. Find the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(the result will be a function of x).

Solution:

If $h \neq 0$ we compute

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\frac{6}{x+h+7} - \frac{6}{x+7}}{h} = \frac{6(x+7)-6(x+h+7)}{h\cdot(x+h+7)\cdot(x+7)} = \frac{-6h}{h\cdot(x+h+7)\cdot(x+7)} \\ &= \frac{-6}{(x+h+7)\cdot(x+7)}. \end{aligned}$$

The last function is rational (in h), thus we get by direct substitution

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-6}{(x+h+7)\cdot(x+7)} = -\frac{6}{(x+7)^2}.$$

For the following exercise, it will be useful to recall that we can rewrite the difference of radicals by multiplying and dividing by the conjugate:

$$\sqrt{a}-\sqrt{b} = \frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}.$$

(2) (6 Points) Consider the function $f(x) = \sqrt{36-x^2}$, whose graph describes the upper semicircle with radius 6 centered at the origin. Let $(a, f(a))$ be any point on the graph of $f(x)$, where $a \neq 0$.

(a) Determine the slope of the line through the origin and the point $(a, f(a))$.

(b) Use facts from high school geometry to compute the slope of the tangent line to the semicircle that passes through the point $(a, f(a))$. Clearly state the fact from geometry that you use to compute the slope.

(c) For any number x in $(-6, 6)$, determine the slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$ and compute its limit L as x approaches a. Find the equation of the line through the point $(a, f(a))$ with slope L. From Section 2.1 we know that this line is the tangent line to the graph of f at the point $(a, f(a))$.
Does the slope L coincide with the slope you computed in (b)? If not, go back and try to find your mistake.

(d) Choose a typical number $a \neq 0$ and plot, on the same set of axes, the graph of f, the line through the origin and the point $(a, f(a))$, and the tangent line.

Solution:

(a) If $a \neq 0$, then the line $\mathcal{L}$ through the origin and the point $(a, f(a))$ has slope

$$\frac{f(a)}{a} = \frac{\sqrt{36-a^2}}{a}.$$

(b) In high school geometry the tangent to the circle at the point $(a, f(a))$ is introduced as the line passing through $(a, f(a))$ and touching the circle. One also learns the fact that the tangent to the circle at the point $(a, f(a))$ is perpendicular to the radius of the circle through $(a, f(a))$.

Since the radius is a segment of the line $\mathcal{L}$ from (a), the slope of the tangent line is the negative reciprocal of the slope we computed in (a), that is,

$$-\frac{a}{\sqrt{36-a^2}}.$$

(c) The slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$ is $\frac{f(x)-f(a)}{x-a}$. Hence the slope of the tangent line is

$$L = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sqrt{36-x^2}-\sqrt{36-a^2}}{x-a},$$

provided this limit exists. Expanding the difference quotient by the conjugate of the numerator and then simplifying we obtain,

$$\begin{aligned}
\frac{\sqrt{36-x^2}-\sqrt{36-a^2}}{x-a} &= \frac{(\sqrt{36-x^2}-\sqrt{36-a^2})\cdot(\sqrt{36-x^2}+\sqrt{36-a^2})}{(x-a)\cdot(\sqrt{36-x^2}+\sqrt{36-a^2})} \\
&= \frac{36-x^2-(36-a^2)}{(x-a)\cdot(\sqrt{36-x^2}+\sqrt{36-a^2})} \\
&= \frac{-x^2+a^2}{(x-a)\cdot(\sqrt{36-x^2}+\sqrt{36-a^2})} \\
&= \frac{-(x-a)(x+a)}{(x-a)\cdot(\sqrt{36-x^2}+\sqrt{36-a^2})} \\
&= \frac{-(x+a)}{\sqrt{36-x^2}+\sqrt{36-a^2}}
\end{aligned}$$

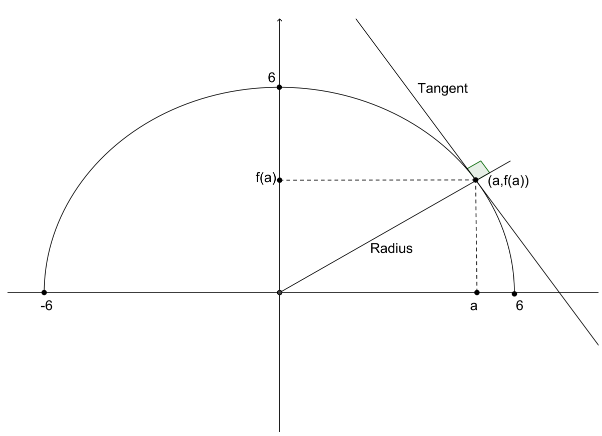
The last function is continuous, thus we obtain the limit by direct substitution

$$L = \lim_{x\to a} \frac{-(x+a)}{\sqrt{36-x^2}+\sqrt{36-a^2}} = \frac{-2a}{2\sqrt{36-a^2}} = -\frac{a}{\sqrt{36-a^2}}.$$

This is the slope of the tangent line and coincides with what we computed in (b). Now the equation of the tangent line is $y - f(a) = L(x-a)$, thus

$$y = -\frac{a}{\sqrt{36-a^2}}\cdot(x-a)+\sqrt{36-a^2}.$$

(d) Sketch:



(3) (1 Point) Write the definition of the derivative of a function f at a point a.

Solution:

The derivative of f at a point a is $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists.

Bonus Problem: (2 Points)

Harry and his sister Sally made the following observation about their ages. When you add up Harry's age and Sally's age, the total is 56 years. So, $H+S=56$. Right now Harry is twice as old as Sally was at the time when he was as old as Sally is now. How old are they?

Solution:

Let's say that T years ago Harry was as old as Sally is now, that is $H-T=S$ or $H=S+T$. Of course, T years ago Sally was $S-T$ years old and we know that Harry is twice as old she was back then, that is, $H=2(S-T)$. Setting equal the last two expressions for H we get $S+T=2(S-T)$, thus $3T=S$. Filling this into $H=2(S-T)$, we obtain $H=4T$ and now the identity $H+S=56$ leads to $7T=56$, thus $T=8$. But then Harry is $H=4T=32$ years old and Sally is $S=3T=24$ years old.

Grading Guidelines:

- Problem (1): (3 points) Two points for simplifying the difference quotient, one for finding the limit.
  Problem (2): (6 points) (a) .5 point. (b) 1 point. (c) 1 point for setting up the correct limit to compute the slope, 1 point for simplifying the difference quotient, 1 point for finding the limit, .5 point for the equation of the tangent line. (d) 1 point for the sketch, which should in particular indicate that the tangent is perpendicular to the radius.
  Problem (3): 1 point. Accept either form of the definition.
  Bonus Problem: 1 point for the correct answer and 1 point for the reasoning.
- Please score in increments of at least 0.5 points. For each part, give full credit only for answers that are both *correct* and *fully explained*.
- Be sure to comment favorably on papers of students who do an unusually good job.
- Take the time to recognize and provide guidance to students who attempt unusual approaches.
- Deductions:
  i) If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point for three or more such mistakes which are not otherwise penalized.
  ii) Deduct one point for unusually messy or poorly organized solutions (at most one or two per paper).