Solution to Assignment 3 for MA 113 - Calculus I

(1) (3 Points) Consider the function $f(x) = 2e^x + ax^4 + 5$, where a is a constant. Determine a such that the tangent line to $y = f(x)$ at the point $(1, f(1))$ has slope 2.

Solution:
Using the differentiation rules we get $f'(x) = 2e^x + 4ax^3$, thus $f'(1) = 2e + 4a$. The tangent line at the point $(1, f(1))$ has slope 2 if and only if $2 = f'(1) = 2e + 4a$. Solving for a we obtain $a = \frac{1-e}{2}$. Hence the tangent line has slope 2 if and only if $a = \frac{1-e}{2}$.

For the following exercise, it might be helpful to recall the quadratic formula. The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity inside the radical, $b^2 - 4ac$, is called the discriminant. It follows that we have two real roots if the discriminant is positive, one real root if the discriminant is 0, and no real roots if the discriminant is negative.

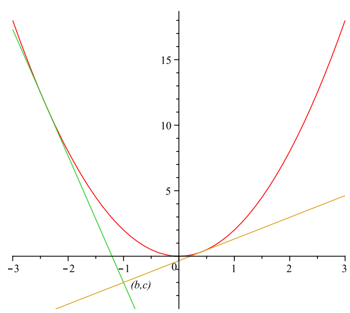
(2) (7 Points) Consider the parabola $y = 2x^2$, and let (b, c) be a point which may or may not lie on the parabola.

(a) Write the equation of the tangent line to the parabola $y = 2x^2$ at $(a, 2a^2)$.

(b) Make a conjecture as to how many tangent lines of the parabola pass through the given point (b, c). How does the answer depend on the point (b, c)? See the picture below for one option.

(c) Give conditions on b and c which tell us whether we have exactly 0, 1 or 2 tangent lines through (b, c).
[Hint: If we require the tangent line in part (a) to pass through point (b, c), we obtain an equation for a. Write out this equation and solve for a. This will give you the desired conditions on b and c.]

(d) Interpret your answers in (c) geometrically. What do these conditions tell us about the location of the point (b, c) with respect to the parabola?

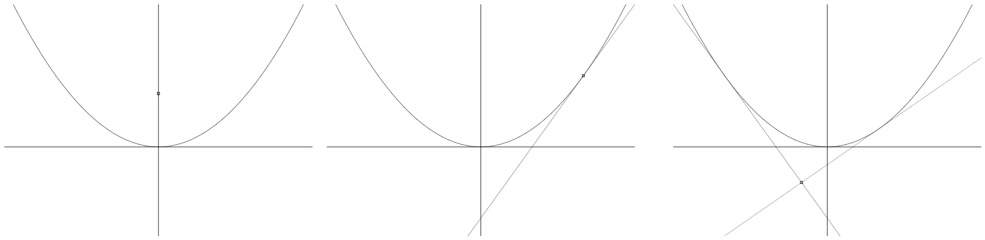


Solution:

(a) The parabola is the graph of the function $f(x) = 2x^2$. The equation of the tangent line to the graph of f at a is $y - f(a) = f'(a)(x - a)$. Using that $f'(x) = 4x$ and thus $f'(a) = 4a$, we obtain for the equation of the tangent line $y - 2a^2 = 4a(x - a)$, or

$$y = 4ax - 2a^2. \tag{1}$$

(b) Studying some examples, see pictures below, one might guess the following: i) if the point lies above the parabola, there are no tangent lines to the parabola that pass through that point; ii) if the point lies on the parabola, there is exactly one tangent line that passes through that point; iii) if the point lies below the parabola, then it appears that we may find two tangent lines that pass through that point.



(c) If the point (b, c) lies on the tangent line to the graph of f at the point (a, a^2), Equation (1) gives

$$c = 4ab - 2a^2, \tag{2}$$

thus a has to satisfy $2a^2 - 4ba + c = 0$ or $a^2 - 2ba + \frac{c}{2} = 0$. Each solution corresponds to a tangent line passing through (b, c). According to the quadratic formula, the solutions (roots) of this quadratic equation in the unknown a are

$$a = b \pm \sqrt{b^2 - \frac{c}{2}}.$$

It follows that Equation (2) has precisely
i) two solutions if $b^2 - \frac{c}{2} > 0$, hence $c < 2b^2$,
ii) one solution (sometimes called a double root) if $b^2 - \frac{c}{2} = 0$, hence $c = 2b^2$,
iii) no solution if $b^2 - \frac{c}{2} < 0$, hence $c > 2b^2$.

(d) Since the solutions of Equation (2) correspond to the tangent lines to the curve $y = 2x^2$ which pass through the point (b, c), the result in part (c) can be rephrased as follows: The number of tangents to the parabola that pass through the given point (b, c) is
i) two if (b, c) lies below the parabola (that is, $c < 2b^2$),
ii) one if (b, c) lies on the parabola (that is, $c = 2b^2$),
iii) zero if (b, c) lies above the parabola (that is, $c > 2b^2$).
Thus, we see that our conjecture from part (b) is correct.

Bonus Problem: (2 Points) How can you plant 12 trees in 6 straight lines such that each line contains exactly 4 trees?

Solution:
Consider a six-pointed star (hexagram or star of David, consisting of two interlocking triangles) and plant the trees at the 6 vertices and at the 6 intersections points.

Grading Guidelines:

- Problem (1): 1 point for the derivative $f'(x)$, 1 point for realizing that $f'(1)$ has to be 2, and 1 point for finding a.
  Problem (2a): 1 point for the derivative, 1 point for the equation of tangent line. The variable a for the point of tangency should be different from x and y, the coordinates we use to describe the line.
  Problem (2b): 1 point for any consistent conjecture. We cannot deduct for guessing wrong.
  Problem (2c): 1 point for the equation, 1 point for the solutions, 1 point for the relation between discriminant and number of solutions.
  Problem (2d): 1 point.
  Bonus Problem: 1 point for the correct answer and 1 point for the reasoning.
- Please score in increments of at least 0.5 points. For each part, give full credit only for answers that are both *correct* and *fully explained.*
- Be sure to comment favorably on papers of students who do an unusually good job.
- Take the time to recognize and provide guidance to students who attempt unusual approaches.
- Deductions:
  i) If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point for three or more such mistakes which are not otherwise penalized.
  ii) Deduct one point for unusually messy or poorly organized solutions (at most one or two per paper).