Solution to Assignment 6 for MA 113 - Calculus I

(1) (6 Points) Carry out the following steps to sketch the graph of the function $f(x) = e^{-x}\sin(x)$ on the interval $[0, 2\pi]$.

(a) Find the critical numbers of f. Compute the local maxima and minima (both coordinates). Give the intervals of increase and decrease.

(b) Find the inflection points of f. Give the intervals where f is concave upward and where f is concave downward.

(c) Make a careful sketch of the graph of f that reflects the above information.

Solution:

(a) Using the product and the chain rule we get for the derivative of f:

$$f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) = e^{-x}\big(\cos(x) - \sin(x)\big).$$

Since $e^{-x} > 0$, we get $f'(x) = 0$ if and only if $\cos(x) - \sin(x) = 0$, which is equivalent to $\cos(x) = \sin(x)$ and thus to $\tan(x) = 1$. This implies $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$. Since f is differentiable on the interval $(0, 2\pi)$, the critical numbers of f inside this interval are $\pi/4$ and $5\pi/4$. In addition, the endpoints 0 and 2π are critical numbers of f.
Furthermore, since f' is continuous, it has constant sign on each of the intervals $(0, \pi/4), (\pi/4, 5\pi/4)$, and $(5\pi/4, 2\pi)$. To determine the sign of f', we compare $\cos(x)$ and $\sin(x)$ and obtain

interval	$(0, \pi/4)$	$(\pi/4, 5\pi/4)$	$(5\pi/4, 2\pi)$
sign of $f'(x)$	+	−	+
I/D	increasing	decreasing	increasing

(Alternatively, we can use sample points and get

interval	$(0, \pi/4)$	$(\pi/4, 5\pi/4)$	$(5\pi/4, 2\pi)$
test point	0.1	1	5
$f'(x)$	≈ 0.81	≈ -0.11	≈ 0.008
I/D	increasing	decreasing	increasing

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as above.)
Hence, f is decreasing on the interval $(\pi/4, 5\pi/4)$, and f is increasing on the intervals $(0, \pi/4)$ and $(5\pi/4, 2\pi)$. Furthermore, the first derivative test for local extrema provides that f has a local maximum at $x = \pi/4$ and a local minimum at $x = 5\pi/4$. The local maximum value at $\pi/4$ is $f(\pi/4) = \frac{\sqrt{2}}{2}e^{-\pi/4} \approx 0.322$ and the local minimum value at $5\pi/4$ is $f(5\pi/4) = -\frac{\sqrt{2}}{2}e^{-5\pi/4} \approx -.0.014$.
(b) For the second derivative of f we get

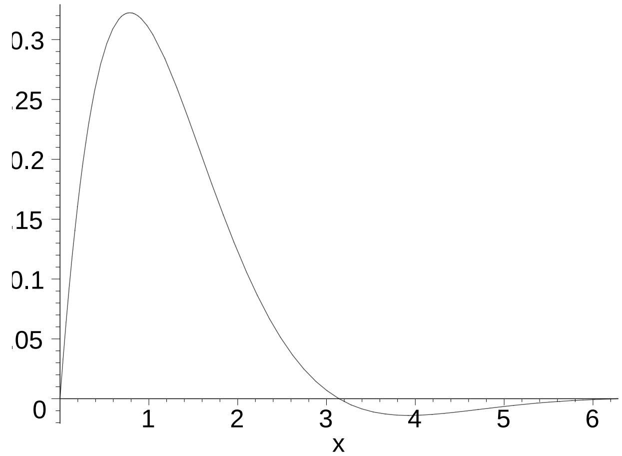
$$f''(x) = -e^{-x}[\cos(x) - \sin(x)] + e^{-x}[-\sin(x) - \cos(x)] = -2e^{-x}\cos(x).$$

Hence $f''(x) = 0$ if and only if $\cos(x) = 0$, that is, if $x = \pi/2$ or $x = 3\pi/2$. The sign of $f''(x)$ equals the sign of $-\cos(x)$. Thus, we obtain the table:

interval	$(0, \pi/2)$	$(\pi/2, 3\pi/2)$	$(3\pi/2, 2\pi)$
sign of $f''(x)$	−	+	−
concavity of f	down	up	down

We conclude that f is concave down on the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$ and that f is concave up on the interval $(\pi/2, 3\pi/2)$. Moreover, the inflection points of f are $(\pi/2, f(\pi/2)) = (\pi/2, e^{-\pi/2})$ and $(3\pi/2, f(3\pi/2)) = (3\pi/2, -e^{-3\pi/2})$.

(c) Graph of f:



(2) (4 Points) Let u and v be two non-negative numbers whose sum is 10. Find the maximum value of u^4v. Determine the values of u and v for which the maximum is attained.

Solution:

We are given $u + v = 10$ with $u, v \geq 0$. It follows that $v = 10 - u \geq 0$, thus u is in the interval $[0, 10]$. We want to find the maximum of $u^4v = u^4(10 - u)$. This means that we are looking for the maximum value of $f(u) = u^4(10 - u) = 10u^4 - u^5$ on the closed interval $[0, 10]$.
To find it we compute the derivative $f'(u) = 40u^3 - 5u^4 = 5u^3(8 - u)$. Solving $f'(u) = 0$ we get that the only critical number of f in the open interval $(0, 10)$ is $u = 8$. Since f is defined on a closed interval, it attains its maximum value at a critical number which also may mean at an endpoint. We compute $f(0) = 0$, $f(10) = 0$, and $f(8) = 8192$. Since 8192 is larger than 0, the maximum value is 8192, and it occurs when $u = 8$ and $v = 10 - 8 = 2$.

Bonus Problem: (2 Points)

A group of professors in the Math department go out for dinner. They have drinks and food and everything, and their agreement is to share the bill evenly. Just before the bill comes, three of the professors go to the bathroom, manage to climb out the window, and leave for good! The bill comes and it is a whopping \$ 120 (including tax and tip). Professor Lovely says: "Those guys walked out on us again! But look. If everyone, in addition to their original share of the bill, throws in an extra two bucks, we can exactly cover the bill." How many people were in the original group?

Solution:

Suppose there were originally x people in the group. So, everybody should've paid \$ $120/x$. Since there are only $x - 3$ people left and Professor Lovely asks each of them to throw in \$ 2 in addition to their share, that will raise $(120/x + 2)(x - 3)$ dollars. Professor Lovely claims that that will exactly cover the bill, that is, $120 = (120/x + 2)(x - 3)$. In order to solve for x we multiply by x, leading to $120x = 120(x - 3) + 2x(x - 3)$, which gives us the quadratic equation $2x^2 - 6x - 360 = 0$ or $x^2 - 3x - 180 = 0$. The solutions are $x = 15$ and $x = -12$. We can ignore the latter one. Hence there were originally 15 people in the group and each of them should've paid $120/15 = 8$ dollars. Since three fellows left the scene, Professor Lovely made the 12 remaining ones pay \$ 10 each.

Acknowledgment: All the puzzlers for the course were taken from the NPR radio show *Car Talk*.

Grading Guidelines:

– Problem (1): (6 points) .5 for f' and .5 for f'', and 1 point for each of the following: intervals of increase/decrease, local extrema, intervals of concave up/concave down, inflection points, graph. (Not mentioning the endpoints as critical numbers goes unpunished.)
  Problem (2): (4 points) 1 point for finding the function (in one variable) to be maximized, 1 point for finding the critical numbers, 1 point for verifying that the absolute maximum is at 8, 1 point for the resulting values

of u and v.
Bonus Problem: 1 point for the answer, 1 point for the reasoning.

- Please score in increments of at least 0.5 points. For each part, give full credit only for answers that are both *correct* and *fully explained*.
- Be sure to comment favorably on papers of students who do an unusually good job.
- Take the time to recognize and provide guidance to students who attempt unusual approaches.
- Deductions:
  i) If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point for three or more such mistakes which are not otherwise penalized.
  ii) Deduct one point for unusually messy or poorly organized solutions (at most one or two per paper).