A behavioral approach to delay-differential systems

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Abstract: We will study linear time-invariant delay-differential systems from the behavioral point of view as it was introduced for dynamical systems by *Willems* (see [22]). It will be presented a ring $\mathcal{H}$ lying between $\mathbb{R}[s,z,z^{-1}]$ and $\mathbb{R}(s)[z,z^{-1}]$, whose elements can be interpreted as a generalized version of delay-differential operators on $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. In this framework, a behavior is the kernel of such an operator. Using the ring $\mathcal{H}$, an algebraic characterization of inclusion resp. equality of the behaviors under consideration is given. Finally, controllability of the behaviors is characterized in terms of the rank of the associated matrices. In the case of time-delay state-space systems this criterion becomes the known Hautus-criterion for spectral controllability.

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1 Introduction

The purpose of this paper is an approach to linear time-invariant delay-differential systems with algebraic methods. In contrast to the work of e. g. *Morse* [16], *Sontag* [21] and more recently *Habets* [8] we will not consider these systems as systems over (polynomial) rings. Instead we will use the behavioral viewpoint for dynamical systems as it was introduced by *Willems* [22]: our objects will be behaviors, which are defined by linear time-invariant delay-differential equations over the time axis $\mathbb{R}$ (for the definition of a behavior see [22]). In the scalar case such equations are given by

$$\sum_{j=0}^{L}\sum_{i=0}^{N} p_{ij} w^{(i)}(t-j) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $p_{ij} \in \mathbb{R}$ and $w^{(i)}$ denotes the i-th derivative of the function w. In our approach only functions w in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ will be considered. In the multivariable case, linear subspaces $\mathcal{B}$ of $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ are investigated, which are the solution space of a system of delay-differential equations, i. e. for which there exist n, L, $N \in \mathbb{N}$ and matrices $P_{ij} \in \mathbb{R}^{n\times m}$ so that

$$\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m) | \textstyle\sum_{j=0}^{L}\sum_{i=0}^{N} P_{ij} w^{(i)}(t-j) = 0,\ t \in \mathbb{R}\}. \tag{1.2}$$

The behavior in (1.2) can be written as $\mathcal{B} = \ker \widetilde{P}$, where $P = \sum_{j=0}^{L}\sum_{i=0}^{N} P_{ij}s^i z^j \in \mathbb{R}[s,z]^{n\times m}$ and $\widetilde{P}$ denotes the associated delay-differential operator from $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ to $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n)$, i. e. $\widetilde{P}w(t) = \sum_{j=0}^{L}\sum_{i=0}^{N} P_{ij} w^{(i)}(t-j)$. Note that (1.2) includes ordinary differential equations ($P \in \mathbb{R}[s]$) as well as the case of a pure delay equation ($P \in \mathbb{R}[z]$). Since the shift yields an isomorphism on $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, it is algebraically more adequate to consider the polynomial ring $\mathbb{R}[s,z,z^{-1}]$ instead of $\mathbb{R}[s,z]$.

Although the space $\mathcal{B}$ is in general infinite-dimensional, via polynomial matrices it is given a description with finitely many parameters. This leads to the possibility of studying special aspects of this type of equations with mainly algebraic methods.

The polynomial approach to time-delay systems was already introduced by *Kamen* [10]. He considered delay-differential operators as special convolution operators in the distributional sense and presented, within this set-up, procedures for the solution of input/output-equations and for the internal description (state-space realizations) of such equations.

In the present paper our starting point will be the solution spaces (or behaviors) $\ker \widetilde{P}$ as given in (1.2). We will not investigate the question, which subspaces of $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ occur as such behaviors. Main ideas for an answer to this question are contained in the thesis of *Soethoudt* [20]. He characterizes behaviors, which have an AR-representation in the purely differential sense. Instead of attacking this (nevertheless interesting) problem of the existence of polynomial representations, we will consider the question of uniqueness: for what pairs of matrices P, Q over $\mathbb{R}[s,z,z^{-1}]$ does hold $\ker \widetilde{P} = \ker \widetilde{Q}$? It should be obvious, that an answer of this question is necessary for the development of a "behavioral theory" using polynomial (AR-) representations for time-delay systems. Simple examples show that the above problem cannot be satisfactorily solved with the help of the ring $\mathbb{R}[s,z,z^{-1}]$ or even $\mathbb{R}(s)[z,z^{-1}]$. The appropriate domain in order to translate relations between behaviors into relations between the associated polynomial matrices lies between these two rings and turns out to be

$$\mathcal{H} = \{p \in \mathbb{R}(s)[z,z^{-1}] \,|\, p(s,e^{-s}) \text{ is an entire function}\}.$$

In the preliminaries an interpretation of the elements of $\mathcal{H}$ as operators on $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is given. It generalizes the interpretation of polynomials in $\mathbb{R}[s,z,z^{-1}]$ as delay-differential operators. Therefore we will refer to these associated operators as delay-differential operators as well.

A similar construction occured already in the work of *Kamen et al.* [11], where the ring Θ generated by the entire functions $(1-e^{-s}e^{\sigma})(s-\sigma)^{-1}$, $\sigma \in \mathbb{C}$, and their derivatives is considered. One can easily see, that the ring $\Theta[s,z]$ in [11, p. 841] is contained in $\mathcal{H}$. *Kamen et al.* also gave an interpretation of the functions $(1-e^{-s}e^{\sigma})(s-\sigma)^{-1}$ as transfer functions of distributed-delay systems.

One main tool in the present approach is the fact that the division properties in the ring $\mathcal{H}$ correspond to the division properties in the ring of entire functions, i. e. for p, $q \in \mathcal{H}$

it holds: p divides q in $\mathcal{H}$ iff $q(s,e^{-s})p(s,e^{-s})^{-1}$ is an entire function. For the associated delay-differential equations this has as a consequence, that it suffices to consider *fundamental solutions*, i. e. functions of the type $w(t)=t^k e^{\lambda t}$ instead of the full solution space. This fits with a result of *Malgrange* [14, p. 318], who proved that the space of all linear combinations of fundamental solutions of a delay-differential equation lies dense in the full space of smooth solutions (with respect to the topology of uniform convergence of all derivatives on all compact subsets in $\mathbb{R}$).

Another important result in our framework is the fact, that $\mathcal{H}$ is a so-called elementary divisor ring. This means firstly, that $\mathcal{H}$ is a Bézout-domain, i. e. every finitely generated ideal in $\mathcal{H}$ is principal. Secondly, every matrix over $\mathcal{H}$ can be brought into diagonal form via multiplication with unimodular matrices from the left and from the right. With this type of normal form (which cannot be achieved e. g. over the ring $\mathbb{R}[s,z,z^{-1}]$), the results for multivariable delay-differential equations can easily be derived from the scalar case.

With this informations about the ring $\mathcal{H}$, which are derived in section 3, we will show in the fourth section how the relations between behaviors as given in (1.2) can be put into correspondence with the division relations of the associated matrices over $\mathcal{H}$. In particular, we prove for $P\in\mathcal{H}^{n\times m}$, $Q\in\mathcal{H}^{r\times m}$: $\ker\widetilde{P}\subseteq\ker\widetilde{Q}$ iff $Q=AP$ for some $A\in\mathcal{H}^{n\times r}$, which yields $\ker\widetilde{P}=\ker\widetilde{Q}$ iff A is unimodular over $\mathcal{H}$.

Finally in the fifth section, controllability of delay-differential systems is considered. In this set-up it is natural to use the notion of controllability for behaviors as introduced by *Willems* [22]. Using a diagonal form for matrices $P\in\mathcal{H}^{n\times m}$, it will be proven that $\ker\widetilde{P}$ is controllable if and only if $\mathrm{rk}_{\mathbb{R}}P(s,e^{-s})=\mathrm{rk}_{\mathcal{H}}P$ for all $s\in\mathbb{C}$. Recently, this characterization has been obtained independently for the same situation of delay-differential equations by *Rocha/Willems* [19]. The given criterion is a generalization of the Hautus-test for time-delay state-space systems, which characterizes the so-called *spectral controllability*, see e. g. *Pandolfi* [18], *Bhat/Koivo* [2], *Manitius/Triggiani* [15], and *Kamen et al.* [11].

2 Preliminaries

In this section we present the framework for our study of delay-differential equations and introduce the notations. Starting with the interpretation of polynomials in $\mathcal{R}$ as delay-differential operators on $\mathcal{C}^\infty(\mathbb{R})$, we first have a glance at the fundamental solutions of the associated equations. This leads us to the corresponding characteristic function and its zeros. Simple examples suggest the introduction of a larger space $\mathcal{H}$ of operators which are closely related to the delay-differential operators. Finally we state the surjectivity of the operators under consideration.

Definition 2.1

a) *Put* $\mathcal{R}:=\mathbb{R}[s,z,z^{-1}]$ *and let* $\mathcal{C}^\infty(\mathbb{R}^m):=\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ *for* $m\geq 1$.

b) *For* $m\geq 1$ *and* $t_0\in\mathbb{R}$ *define the shift* $\sigma^{t_0}:\mathcal{C}^\infty(\mathbb{R}^m)\to\mathcal{C}^\infty(\mathbb{R}^m)$ *by* $(\sigma^{t_0}w)(t)=w(t-t_0)$ *for* $w\in\mathcal{C}^\infty(\mathbb{R}^m)$. *In particular, let* $\sigma:=\sigma^1$.

c) *With* $P=\sum_{j=l}^{L}\sum_{i=0}^{N}P_{ij}s^iz^j\in\mathcal{R}^{n\times m}$ *associate the following delay-differential operator*

$$\begin{array}{rccl}\widetilde{P}: & \mathcal{C}^\infty(\mathbb{R}^m) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^n)\\ & w & \longmapsto & \sum_{j=l}^{L}\sum_{i=0}^{N}P_{ij}\sigma^j w^{(i)},\end{array}\qquad(2.1)$$

where $w^{(i)}=\frac{d^i}{dt^i}w$.

d) *For* $p=\sum_{i=0}^{N}p_is^i\in\mathbb{R}[s]$ *and* $w\in\mathcal{C}^\infty[a,b]$ *we use analogously the notion* $\widetilde{p}(w)(t)=\sum_{i=0}^{N}p_iw^{(i)}(t)$, *hence* $\widetilde{p}(w)\in\mathcal{C}^\infty[a,b]$.

Note that part c) makes indeed sense, since on $\mathcal{C}^\infty(\mathbb{R})$ the operators σ and $\frac{d}{dt}$ commute.

In this context, the solution space in $\mathcal{C}^\infty(\mathbb{R})$ of the scalar equation (1.1) is just $\ker\widetilde{p}$, a linear *shift-invariant* subspace of $\mathcal{C}^\infty(\mathbb{R})$, i. e. $\sigma^t(\ker\widetilde{p})=\ker\widetilde{p}$ for all $t\in\mathbb{R}$. In this section we will only study the scalar equation (1.1). We will come to the multivariable situation in section 4.

Remark 2.2 The map

$$\begin{array}{rccl}T: & \mathcal{R} & \longrightarrow & \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}))\\ & p & \longmapsto & \widetilde{p}\end{array}$$

is an injective algebra-homomorphism. The homomorphism properties $\widetilde{p+q}=\widetilde{p}+\widetilde{q}$, $\widetilde{pq}=\widetilde{p}\circ\widetilde{q}$ can easily be verified. To prove injectivity of T, let $p=\sum_{i,j}p_{ij}s^iz^j\in\mathcal{R}$ and assume that $\widetilde{p}=0$. Then for arbitrary $\lambda\in\mathbb{C}$ and $w\in\mathcal{C}^\infty(\mathbb{R})$ with $w(t)=e^{\lambda t}$ we obtain $0=\widetilde{p}(w)(t)=\sum_{i,j}p_{ij}\lambda^ie^{\lambda(t-j)}=\sum_{i,j}p_{ij}\lambda^ie^{-\lambda j}e^{\lambda t}$ for all $t\in\mathbb{R}$, hence $\sum_{i,j}p_{ij}\lambda^ie^{-\lambda j}=0$. Since this holds true for all $\lambda\in\mathbb{C}$, the linear independence of the functions $\lambda\mapsto\lambda^ie^{\lambda j}$ yields in fact $p_{ij}=0$ for all i, j.

One question we want to attack in this paper is, how to characterize the inclusion $\ker\widetilde{p}\subseteq\ker\widetilde{q}$ in terms of the elements $p,\ q\in\mathcal{R}$. Let us first have a look at a simple

Example 2.3

a) For $p,\ q\in\mathbb{R}[s]\subset\mathcal{R}$ the theory of ordinary differential equations leads to $\ker\widetilde{p}\subseteq\ker\widetilde{q}$ iff p divides q in $\mathbb{R}[s]$, hence iff p divides q in $\mathcal{R}$.

b) It is easily seen that $\ker\widetilde{s}=\{\text{constants}\}\subset\ker\widetilde{z-1}=\{w\in\mathcal{C}^\infty(\mathbb{R})|w\text{ is of period }1\}$. But s does not divide $z-1$ in $\mathcal{R}$. Of course, s divides $z-1$ in $\mathbb{R}(s)[z,z^{-1}]$.

The above shows, that the division properties of the two rings $\mathcal{R}$ and $\mathbb{R}(s)[z,z^{-1}]$ are not useful in the algebraic description of $\ker\widetilde{p}\subseteq\ker\widetilde{q}$.

As with ordinary differential equations, some more information about the solution space of (1.1) is obtainable by studying fundamental solutions $w(t)=t^ke^{\lambda t}$ where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. In the present case this leads to the characteristic function of (1.1), which will be an entire function. We will need the concept of a characteristic function in a slightly more general situation, handled in the next definition. In the special case of part b) of the definition, these functions are often called quasi-polynomials (see e. g. [7, p. 63]) or exponential polynomials (see [1, ch. 12]). In part c) and d) we introduce some notations useful for the sequel.

Definition 2.4

a) *For* $p = \sum_{j=l}^{L} p_j z^j \in \mathbb{R}(s)[z, z^{-1}]$ *with* $p_j \in \mathbb{R}(s)$ *and* $p_l \neq 0 \neq p_L$ *define the degree of* p *to be* $\deg_z p := L - l$. *Further let*

$$p^*(s) := \sum_{j=l}^{L} p_j(s) e^{-js} \text{ for all } s \in \mathbb{C} \text{ not being a pole of } p_j,\ j = l \ldots, L.$$

Then $p^* \in M(\mathbb{C})$, *the set of all meromorphic functions on* $\mathbb{C}$.

b) *If* $p = \sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} s^i z^j \in \mathcal{R}$, *then* $p^* \in H(\mathbb{C})$, *the set of entire functions.* p^* *is called the characteristic function of the delay-differential equation*

$$\sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} w^{(i)}(t - j) = 0,\ t \in \mathbb{R}.$$

c) *For* $f \in M(\mathbb{C})$ *and* $\alpha \in \mathbb{C}$ *denote the order of the zero (resp. pole)* α *of* f *by*

$$\mu_\alpha(f) := \min\{k \in \mathbb{Z} \,|\, (s - \alpha)^{-k} f \text{ holomorphic and not zero around } \alpha\}.$$

d) *For* $f_1, \ldots, f_r \in M(\mathbb{C})$ *let* $\mathcal{V}(f_1, \ldots, f_r) = \{\alpha \in \mathbb{C} \,|\, \mu_\alpha(f_i) \geq 1,\ i = 1, \ldots, r\}$ *be the set of common zeros of* $f_1, \ldots, f_r$.

Note that we interpret here s as algebraic indeterminate over $\mathbb{R}$ as well as complex variable.

Remark 2.5 The map $\mathbb{R}(s)[z, z^{-1}] \to M(\mathbb{C}),\ p \mapsto p^*$ is an injective ring homomorphism. The injectivity follows, as in Remark 2.2, from the linear independence of the functions $s \mapsto s^k e^{js}$.

With this notation, we get from the theory of delay-differential equations for $p \in \mathcal{R}$ and for the function $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}),\ w(t) = t^k e^{\lambda t}$

$$w \in \ker \widetilde{p} \iff \mu_\lambda(p^*) > k \tag{2.2}$$

(see [1, p. 54/55] for a special case). This can also be proven directly by showing that $\widetilde{p}w(t) = \frac{d^k}{ds^k}(p^*(s)e^{st})|_{s=\lambda}$. As with ordinary differential equations it is true that with $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ also $\operatorname{Re} w, \operatorname{Im} w \in \mathcal{C}^\infty(\mathbb{R})$ are in $\ker \widetilde{p}$.

The foregoing consideration indicates, that a first knowledge about the dimension of $\ker \widetilde{p}$ can be obtained by calculating the number of zeros of the associated characteristic function p^*. Using the theory of entire functions this can be done in the following sense.

Proposition 2.6 *Let* $p \in \mathcal{R}$. *Then*

$$\#\mathcal{V}(p^*) < \infty \iff p = z^k \phi \text{ for some } k \in \mathbb{Z} \text{ and } \phi \in \mathbb{R}[s] \backslash \{0\}.$$

This result can be proven by use of some facts about the order of entire functions, as they can be found e. g. in [9]. Since we are not aware of an explicit proof in the literature, we present here a short sketch, how to establish the result with the help of [9].

PROOF: "$\Leftarrow$" is obvious.

"$\Rightarrow$" Let $p = \sum_{j=l}^{L} p_j z^j \in \mathcal{R}$ with $p_j \in \mathbb{R}[s]$. If $\#\mathcal{V}(p^*) < \infty$, then $p^* = ae^g$ with $a \in \mathbb{C}[s]$ and $g \in H(\mathbb{C})$. Suppose that g is not a constant. From [9, 2.7.3, 2.7.4, 4.2.1] it follows $\operatorname{ord}(p^*) = \operatorname{ord}(\sum_{j=l}^{L} p_j e^{-j\cdot}) \leq 1$, where the order $\operatorname{ord}(f)$ of an entire function f is defined as in [9, 1.11.1]. But then [9, 2.7.3, 2.7.5] implies $g \in \mathbb{C}[s]$ and moreover $g(s) = \alpha s + \beta$ with some α, $\beta \in \mathbb{C}$. Hence $p^*(s) = \sum_{j=l}^{L} p_j(s) e^{-js} = a(s)e^{\beta}e^{\alpha s}$. Now, from the independence of the functions $s^k e^{\alpha s}$ we get $\alpha \in \{-L, \ldots, -l\}$ and $p_j = 0$ for $j \neq -\alpha$. Thus $p = p_{-\alpha} z^{\alpha}$. □

Note the simple fact, that for $p = z^k \phi \in \mathcal{R}$ with $\phi \in \mathbb{R}[s]$ and $k \in \mathbb{Z}$ one has $\ker \widetilde{p} = \ker \widetilde{\phi}$, which is just the solution space of an ordinary linear homogeneous differential equation with constant coefficients over $\mathbb{R}$. Hence, as an immediate consequence of (2.2) and Prop. 2.6 we get

$$\#\mathcal{V}(p^*) = \infty \iff \dim \ker \widetilde{p} = \infty$$

for arbitrary $p \in \mathcal{R}$. In other words, $\ker \widetilde{p}$ is finite-dimensional if and only if $\widetilde{p}$ is a (shifted) ordinary differential operator. Moreover, for $q \in \mathcal{R}$ and $\phi \in \mathbb{R}[s]\backslash\{0\}$ the finite dimensionality of $\ker \widetilde{\phi}$ together with (2.2) implies the crucial fact

$$\frac{q^*}{\phi} \in H(\mathbb{C}) \iff \ker \widetilde{\phi} \subseteq \ker \widetilde{q}. \tag{2.3}$$

This easy equivalence is central for our framework, as it allows us to introduce a bigger class $\mathcal{H}$ of linear operators on $\mathcal{C}^{\infty}(\mathbb{R})$, which are closely related to delay-differential-operators. More precisely, for $p = q\phi^{-1} \in \mathbb{R}(s)[z, z^{-1}]$, where $p^* = q^*\phi^{-1} \in H(\mathbb{C})$, it is possible to define $\widetilde{p} = \widetilde{q} \circ \widetilde{\phi}^{-1}$.

We introduce precisely these objects in the following definition and show their well-definedness as well as some elementary properties afterwards in Remark 2.8.

Definition 2.7

a) *Put* $\mathcal{H} := \{p \in \mathbb{R}(s)[z, z^{-1}] \mid p^* \in H(\mathbb{C})\}$.

b) *For* $p = q\phi^{-1} \in \mathcal{H}$ *with* $q \in \mathcal{R}$ *and* $\phi \in \mathbb{R}[s]\backslash\{0\}$ *define the operator*

$$\begin{array}{rccl} \widetilde{p}: & \mathcal{C}^{\infty}(\mathbb{R}) & \longrightarrow & \mathcal{C}^{\infty}(\mathbb{R}) \\ & w & \longmapsto & \widetilde{p}(w) := \widetilde{q}(v), \text{ where } v \in \mathcal{C}^{\infty}(\mathbb{R}) \text{ with } \widetilde{\phi}(v) = w. \end{array}$$

We call $\widetilde{p}$ *a delay-differential operator also if* $p \in \mathcal{H}$.

Remark 2.8

1) From Remark 2.5 it follows that $\mathcal{H}$ is a commutative domain.
2) One has to establish the well-definedness of the map $\widetilde{p}$. First, for fixed $q \in \mathcal{R}$ and $\phi \in \mathbb{R}[s]$ with $q\phi^{-1} \in \mathcal{H}$ the well-definedness of the map $w \mapsto \widetilde{q}(v)$, where $v \in \mathcal{C}^{\infty}(\mathbb{R})$ satisfies $\widetilde{\phi}(v) = w$ is a consequence of (2.3). Next, to see that the map $\widetilde{p}$ does not depend on the special representation of p, let $p = q\phi^{-1} = q'\psi^{-1} \in \mathcal{H}$. For $w \in \mathcal{C}^{\infty}(\mathbb{R})$ put $\widetilde{\phi}(v) = w = \widetilde{\psi}(v')$ and $\widetilde{\phi}(h) = v'$ with suitable v, v', $h \in \mathcal{C}^{\infty}(\mathbb{R})$. Then $\widetilde{\psi}(h) - v \in \ker \widetilde{\phi} \subseteq \ker \widetilde{q}$ and therefore $\widetilde{q}(v) = \widetilde{q}(\widetilde{\psi}(h)) = \widetilde{q'}(\widetilde{\phi}(h)) = \widetilde{q'}(v')$.

3) It can easily be verified that $\widetilde{p}$ is an endomorphism on $\mathcal{C}^\infty(\mathbb{R})$. Moreover, the ring $\mathcal{H}$ can be viewed as a subring of $\mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}))$. To see this, we need to prove that the map $p \mapsto \widetilde{p}$ is an injective ring homomorphism. For this, let $p = a\phi^{-1}$, $q = b\psi^{-1} \in \mathcal{H}$ with $a, b \in \mathcal{R}$ and $\phi, \psi \in \mathbb{R}[s]$. For $w \in \mathcal{C}^\infty(\mathbb{R})$ define $v \in \mathcal{C}^\infty(\mathbb{R})$ such that $\widetilde{\phi\psi}(v) = w$. Then $\widetilde{p+q}(w) = (\widetilde{a\psi + b\phi})(v) = \widetilde{a\psi}(v) + \widetilde{b\phi}(v) = \widetilde{p}(w) + \widetilde{q}(w)$ and from $\widetilde{\psi}(\widetilde{\phi}(v)) = w$ it follows $\widetilde{p} \circ \widetilde{q}(w) = \widetilde{p}(\widetilde{b}(\widetilde{\phi}(v))) = \widetilde{p} \circ \widetilde{\phi}(\widetilde{b}(v)) = \widetilde{a} \circ \widetilde{b}(v) = \widetilde{ab}(v) = \widetilde{pq}(w)$, where we used the homomorphism properties of T as defined in Remark 2.2. The injectivity of $p \mapsto \widetilde{p}$ follows from the same remark.

4) A special case of the homomorphism property of $p \mapsto \widetilde{p}$ is the following: from $p = q\phi^{-1} \in \mathcal{H}$ one has obviously $p\phi = q = \phi p$ in the ring $\mathcal{H}$. The definition of $\widetilde{p}$ tells us that $\widetilde{q}(v) = \widetilde{p} \circ \widetilde{\phi}(v)$ for all $v \in \mathcal{C}^\infty(\mathbb{R})$ and $\widetilde{q}(w) = \widetilde{q}(\widetilde{\phi}(v)) = \widetilde{\phi}(\widetilde{q}(v)) = \widetilde{\phi} \circ \widetilde{p}(w)$ for $v, w \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\widetilde{\phi}(v) = w$. Hence it is indeed $\widetilde{q} = \widetilde{p} \circ \widetilde{\phi} = \widetilde{\phi} \circ \widetilde{p}$.

This shows that Def. 2.7 b) represents the unique extension of the algebra-homomorphism T given in Remark 2.2 from $\mathcal{R}$ to the larger ring $\mathcal{H}$.

Let us illustrate the general delay-differential operator by the following example, which is in some sense the simplest non-ordinary delay-differential operator.

Example 2.9 Let $p := (z-1)s^{-1} \in \mathbb{R}(s)[z]$. Then $p^*(s) = (e^{-s} - 1)s^{-1}$ is an entire function, thus $p \in \mathcal{H}$. The associated operator is given by

$$\begin{array}{rcl} \widetilde{p} : \mathcal{C}^\infty(\mathbb{R}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}) \\ w & \longmapsto & \sigma(v) - v, \text{ where } v^{(1)} = w. \end{array}$$

Obviously, $\ker \widetilde{(z-1)} = \{v \in \mathcal{C}^\infty(\mathbb{R}) | v \text{ is of period } 1\}$, therefore $\ker \widetilde{p} = \{w \in \mathcal{C}^\infty(\mathbb{R}) | \exists v \in \mathcal{C}^\infty(\mathbb{R}) \text{ of period 1 and with } v^{(1)} = w\}$, which is a proper subspace of $\ker \widetilde{(z-1)}$. Note that in the above case we have $\widetilde{p}(w) = \int_t^{t-1} w(\tau)d\tau$, which indicates, that $\mathcal{H}$ includes not only point-delay but also distributed-delay operators.

As we will see in section 4, it is just the ring $\mathcal{H}$, which gives an algebraic description of the relation between behaviors of the type $\ker \widetilde{p} \subset \mathcal{C}^\infty(\mathbb{R})$: the lattice of kernels of operators $\widetilde{p}$ corresponds to the lattice of principal ideals in $\mathcal{H}$. Therefore, for the development of this correspondence it makes sense, to consider also delay-differential operators in the generalized version of Def. 2.7. The ring $\mathcal{H}$ will be investigated in the next section.

We close the preliminaries with the following

Proposition 2.10 *Let $p \in \mathcal{H} \backslash \{0\}$. Then:*

a) The map $\widetilde{p} \in \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}))$ is surjective.

b) Let $\deg_z p = L > 0$. If $w \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\widetilde{p}(w) = 0$ and $w|_{[k,k+L]} = 0$ for some $k \in \mathbb{Z}$, then $w = 0$.

The result of part a) can be found in [6, p. 697]. Since [6] uses rather difficult methods to prove surjectivity also for other (more general) operators, we present a complete and elementary proof of both parts of the proposition in the appendix. Of course, the surjectivity of $\widetilde{p}$ is well-known if $p \in \mathbb{R}[s]$.

3 Properties of the ring $\mathcal{H}$

Beside others, two facts about the ring $\mathcal{H}$ will be important for the sequel. The one is, that the division structure of $\mathcal{H}$ corresponds to the division properties of the associated entire functions in the full ring $H(\mathbb{C})$. This is made precise in part e) of Prop. 3.1. The other main fact about $\mathcal{H}$ is its advantageous ring structure. In Thm. 3.2 we will show that $\mathcal{H}$ is a *Bézout-ring*, i. e. that every finitely generated ideal is principal. Stated in other words, finitely many elements $p_1, \dots, p_r \in \mathcal{H}$ have a greatest common divisor $d \in \mathcal{H}$, which fulfills a Bézout-equation $d = \sum_{i=1}^{r} a_i p_i$ over $\mathcal{H}$. Furthermore, with Lemma 3.4 it will be proven that $\mathcal{H}$ is an *elementary divisor ring*, which means that matrices over $\mathcal{H}$ can be brought into diagonal form via multiplication with unimodular matrices from both sides. This is a very useful fact in order to handle the matrix-case of delay-differential equations. One should note that both properties hold true also for the ring $H(\mathbb{C})$, see e. g. [17, Thm. 5, p. 136 and Thm. 8, p. 141], but not for $\mathcal{R}$.

Proposition 3.1

a) If $p \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, then $p^*(\bar{\alpha}) = \overline{p^*(\alpha)}$, where $^-$ denotes complex conjugation.

b) Define $\mathcal{H}^\times := \{p \in \mathcal{H} \,|\, p \text{ is a unit}\}$.
Then $\mathcal{H}^\times = \{az^k \,|\, a \in \mathbb{R}\backslash\{0\},\, k \in \mathbb{Z}\} = \{p \in \mathcal{H} \,|\, \mathcal{V}(p^*) = \emptyset\}$.

c) $\mathcal{H}$ is not a unique factorization domain and not a Noetherian ring.

d) For $p \in \mathcal{H}$ the following statements are equivalent: i) p is irreducible, ii) $p = \phi z^k$ for some irreducible $\phi \in \mathbb{R}[s]$ and $k \in \mathbb{Z}$, iii) p is prime.

e) Let $p, q \in \mathcal{H}$. Then $p^* \,|\, q^*$ in $H(\mathbb{C}) \Longleftrightarrow p \,|\, q$ in $\mathcal{H}$.

f) For $p, q \in \mathcal{H}$, not both zero, there exists a greatest common divisor (gcd) $d \in \mathcal{H}\backslash\{0\}$ of p, q, which is unique up to multiplication by units in $\mathcal{H}$. Moreover, $\mathcal{V}(d^*) = \mathcal{V}(p^*, q^*)$. In particular, p and q are coprime in $\mathcal{H}$ if and only if $\mathcal{V}(p^*, q^*) = \emptyset$.

g) Let $p = ad$, $q = bd \in \mathcal{H}\backslash\{0\}$ with d being a gcd of p, q and $a, b \in \mathcal{H}$. Then $c := abd \in \mathcal{H}$ is a least common multiple (lcm) of p, q. A lcm is unique up to multiplication by units in $\mathcal{H}$.

PROOF: a) is obvious.

b) Let $p \in \mathcal{H}^\times$, then p is also a unit in $\mathbb{R}(s)[z, z^{-1}]$, thus $p = az^k$ for some $a \in \mathbb{R}(s)$ and $k \in \mathbb{Z}$. Since $p^*(s) = a(s)e^{-ks}$ and $(p^{-1})^*(s) = a(s)^{-1}e^{ks}$ are both entire functions, it follows $a \in \mathbb{R}\backslash\{0\}$. The last equality holds with Prop. 2.6.

c) Consider $z - 1 \in \mathcal{H}$. Let $(\alpha_i)_{i\in\mathbb{N}} \subset \mathbb{C}\backslash\{0\}$ so that $e^{-\alpha_i} - 1 = 0$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and $\alpha_{2i+1} = \overline{\alpha_{2i}}$ for $i \in \mathbb{N}$. Then $p_i := (s - \alpha_{2i})(s - \alpha_{2i+1}) \in \mathbb{R}[s]$ satisfies $z - 1 = \frac{z-1}{p_i} p_i = \frac{z-1}{\prod_{i=1}^{n} p_i} \prod_{i=1}^{n} p_i$ and these are factorizations of $z - 1$ in $\mathcal{H}$. Moreover, the chain

$$\tfrac{z-1}{p_1}\mathcal{H} \subseteq \tfrac{z-1}{p_1 p_2}\mathcal{H} \subseteq \tfrac{z-1}{p_1 p_2 p_3}\mathcal{H} \subseteq \dots$$

of ideals in $\mathcal{H}$ will not become stationary.

d) "i) $\Rightarrow$ ii)" Let $p \in \mathcal{H}$ be irreducible. By b) there exists $\alpha \in \mathbb{C}$ with $p^*(\alpha) = 0$. If $\alpha \in \mathbb{R}$, then $p = \frac{p}{s-\alpha}(s - \alpha)$ is a factorization in $\mathcal{H}$, thus $\frac{p}{s-\alpha}$ has to be a unit in $\mathcal{H}$. By b) this

yields $p = az^k(s-\alpha)$ for some non-zero $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, which gives ii). If $\alpha \notin \mathbb{R}$, then with a) one gets analogously $p = az^k(s-\alpha)(s-\bar{\alpha})$.
"ii) $\Rightarrow$ iii)" Let $\phi \in \mathbb{R}[s]$ be irreducible. Then ϕ is prime in $\mathbb{R}[s]$ and of the form $\phi = s-\alpha$ or $\phi = (s-\alpha)(s-\bar{\alpha})$. Suppose $p = \phi z^k$ and $p\,|\,fg$ in $\mathcal{H}$ for some $f, g \in \mathcal{H}$. Then $(fg)^* p^{*^{-1}} = (f^*g^*)p^{*^{-1}} \in H(\mathbb{C})$ and both cases for ϕ imply by use of a): $p\,|\,f$ or $p\,|\,g$.
"iii) $\Rightarrow$ i)" holds true in every commutative domain.

e) The direction "$\Leftarrow$" holds since $p \mapsto p^*$ is a ring homomorphism. "$\Rightarrow$" Let $q^*(p^*)^{-1} \in H(\mathbb{C})$. In the field $\mathbb{R}(s,z)$ we can write $qp^{-1} = ab^{-1}$ with coprime $a, b \in \mathbb{R}[s,z]$. The Theorem of Bézout for algebraic curves implies $\#\{(\lambda,\mu) \in \mathbb{C}^2\,|\,a(\lambda,\mu) = 0 = b(\lambda,\mu)\} < \infty$. Since $a^*(b^*)^{-1} = q^*(p^*)^{-1} \in H(\mathbb{C})$ yields $\mathcal{V}(b^*) \subseteq \mathcal{V}(a^*)$, we get $\#\mathcal{V}(b^*) < \infty$. By use of Prop.2.6 this leads to $b = \phi z^k$ for some $\phi \in \mathbb{R}[s]\backslash\{0\}$ and $k \in \mathbb{Z}$. Hence $qp^{-1} = az^{-k}\phi^{-1} \in \mathcal{H}$.

f) Since $\mathcal{H} \subset \mathbb{R}(s)[z,z^{-1}]$, there exists a gcd $d \in \mathbb{R}(s)[z,z^{-1}]$ of p, q. Thus $p = fd$, $q = gd$ with coprime $f, g \in \mathbb{R}(s)[z,z^{-1}]$.
In order to derive from this suitable factorizations in $\mathcal{H}$, we shall shift the poles of f^* or g^* and the common zeros of f^* and g^* within multiplicities into the factor d. To do so, let

$$\mathcal{P} = \{\alpha \in \mathbb{C} | \mu_\alpha(f^*) < 0 \text{ or } \mu_\alpha(g^*) < 0\}$$

be the set of poles of f or g. Then we have $\#\mathcal{P} < \infty$ as well as $\#\mathcal{V}(f^*,g^*) < \infty$ and $\mathcal{P} \cap \mathcal{V}(f^*,g^*) = \emptyset$. Put

$$\begin{aligned} \phi &:= \prod_{\alpha\in\mathcal{P}} (s-\alpha)^{\max\{-\mu_\alpha(f^*),-\mu_\alpha(g^*)\}} \in \mathbb{R}[s], \\ \psi &:= \prod_{\alpha\in\mathcal{V}(f^*,g^*)} (s-\alpha)^{\min\{\mu_\alpha(f^*),\mu_\alpha(g^*)\}} \in \mathbb{R}[s]. \end{aligned}$$

This leads to

$$p = \frac{f\phi}{\psi}\frac{\psi}{\phi}d,\ q = \frac{g\phi}{\psi}\frac{\psi}{\phi}d \ \text{ where } \frac{f\phi}{\psi},\ \frac{g\phi}{\psi} \in \mathcal{H} \ \text{ and } \mathcal{V}\Big((\frac{f\phi}{\psi})^*, (\frac{g\phi}{\psi})^*\Big) = \emptyset. \tag{3.1}$$

Moreover, $\frac{\psi}{\phi}d \in \mathcal{H}$, for if $\alpha \in \mathbb{C}$ would be a pole of $(\frac{\psi}{\phi}d)^*$, then it would follow $\alpha \in \mathcal{V}((\frac{f\phi}{\psi})^*, (\frac{g\phi}{\psi})^*)$ since $p^*, q^* \in H(\mathbb{C})$. Hence we have a factorization $p = f'd'$, $q = g'd'$ in $\mathcal{H}$ and $\mathcal{V}((f')^*,(g')^*) = \emptyset$ implies that $(d')^*$ is a gcd of p^*, q^* in $H(\mathbb{C})$.
To show that d' is a gcd of p, q in $\mathcal{H}$, let $p = f''d''$, $q = g''d''$ with f'', g'', $d'' \in \mathcal{H}$. Then $p^* = (f'')^*(d'')^*$, $q^* = (g'')^*(d'')^*$ and thus $(d'')^*\,|\,(d')^*$ in $H(\mathbb{C})$. By e) this yields $ad'' = d'$ for some $a \in \mathcal{H}$ and therefore d' is a gcd of p, q in $\mathcal{H}$. This argument also implies the uniqueness property claimed for a gcd in $\mathcal{H}$.
The equality $\mathcal{V}(d^*) = \mathcal{V}(p^*,q^*)$ follows from (3.1) and the last claim of f) is an easy consequence of b).

g) Obviously $p\,|\,c$ and $q\,|\,c$ in $\mathcal{H}$. Let $c' \in \mathcal{H}$ be another common multiple of p and q, i. e. let there exist $v, w \in \mathcal{H}$ with $adv = c' = bdw$. Therefore $av = bw$ and $a^*v^* = b^*w^*$ in $H(\mathbb{C})$. This yields $w^* = (a^*v^*)(b^*)^{-1} \in H(\mathbb{C})$ and moreover $v^*(b^*)^{-1} \in H(\mathbb{C})$, since by f) a^* and b^* have no common zeros. From e) we get the existence of $b' \in \mathcal{H}$ with $bb' = v$ and thus $c' = adbb' = cb'$. □

Now we can prove

Theorem 3.2 *$\mathcal{H}$ is a Bézout-ring, i. e. every finitely generated ideal is a principal ideal.*

PROOF: We need to show that for p, $q \in \mathcal{H}$ and a gcd $d \in \mathcal{H}$ of p, q there exist a, $b \in \mathcal{H}$ so that $d = ap + bq$, for this implies $p\mathcal{H} + q\mathcal{H} = d\mathcal{H}$. Without loss of generality we can assume $d = 1$, hence by Prop. 3.1 f) that $\mathcal{V}(p^*, q^*) = \emptyset$.

Step 1) The elements p, q are coprime also in $\mathbb{R}(s)[z, z^{-1}]$.
To see this, let $uv = p$, $uw = q$ with u, v, $w \in \mathbb{R}(s)[z, z^{-1}]$, then let $u = \widetilde{u}\phi^{-1}$, $v = \widetilde{v}\psi^{-1}$ with $\widetilde{u}$, $\widetilde{v} \in \mathcal{R}$, ϕ, $\psi \in \mathbb{R}[s]$ and where both $\widetilde{u}$, ϕ as well as $\widetilde{v}$, ψ are coprime pairs in $\mathcal{R}$. Then $\widetilde{u}\widetilde{v} = p\phi\psi$ and $\deg_z \widetilde{u} \geq 1$ would imply, that all irreducible factors u_i of $\widetilde{u}$ with $\deg_z u_i \geq 1$ divide p in $\mathcal{R}$. Similarly $u_i \,|\, q$ in $\mathcal{R}$, which contradicts the coprimeness of p, q in $\mathcal{H}$. Thus $u \in \mathbb{R}(s)$ and is therefore a unit in $\mathbb{R}(s)[z, z^{-1}]$.

Hence there exists a Bézout-equation in $\mathbb{R}(s)[z, z^{-1}]$, i. e.

$$1 = ap + bq \text{ with suitable } a,\, b \in \mathbb{R}(s)[z, z^{-1}]. \tag{3.2}$$

Step 2) Next we will vary the coefficients a, b of (3.2) in such a way, that we get a Bézout-equation for p and q with coefficients in $\mathcal{H}$. More precisely, we will construct a rational function $v \in \mathbb{R}(s)$ so that

$$b + vp,\ a - vq \in \mathcal{H}. \tag{3.3}$$

Then (3.2) will imply the Bézout-equation $1 = (a - vq)p + (b + vp)q$ in $\mathcal{H}$.

Step 2a) In order to achieve (3.3) we have to get rid of the poles of a^* and b^*. Therefore write

$$a = \frac{\widetilde{a}}{\psi},\ b = \frac{\widetilde{b}}{\phi} \text{ with } \widetilde{a},\, \widetilde{b} \in \mathcal{H},\ \psi,\, \phi \in \mathbb{R}[s] \text{ and } \mathcal{V}(\widetilde{a}^*, \psi) = \mathcal{V}(\widetilde{b}^*, \phi) = \emptyset. \tag{3.4}$$

Let $h \in \mathbb{R}[s]$ be a gcd of ψ, ϕ and $\psi = h\psi_1$, $\phi = h\phi_1$ with ψ_1, $\phi_1 \in \mathbb{R}[s]$. Then (3.2) becomes

$$h\psi_1\phi_1 = \phi_1\widetilde{a}p + \psi_1\widetilde{b}q \tag{3.5}$$

where all elements are in $\mathcal{H}$. From $\psi_1(h\phi_1 - \widetilde{b}q) = \phi_1\widetilde{a}p$ and $\mathcal{V}(\psi_1, \phi_1) = \emptyset = \mathcal{V}(\widetilde{a}^*, \psi_1)$ it results with Prop. 3.1 e) $\psi_1 \,|\, p$ in $\mathcal{H}$. So let $p = p_1\psi_1$ with $p_1 \in \mathcal{H}$. Similarly it is $q = q_1\phi_1$ with $q_1 \in \mathcal{H}$. Thus, after cancellation of $\psi_1\phi_1$, (3.5) reads

$$h = \widetilde{a}p_1 + \widetilde{b}q_1. \tag{3.6}$$

Step 2b) Put $v = \frac{f}{h\psi_1\phi_1} \in \mathbb{R}(s)$, where $f \in \mathbb{R}[s]$ still has to be specified. Then (3.3) implies that we have to find $f \in \mathbb{R}[s]$ such that

$$\left.\begin{array}{rcl} (b + vp)^* & = & \left(\frac{\widetilde{b}}{h\phi_1} + \frac{f}{h\phi_1\psi_1}p_1\psi_1\right)^* = \frac{(\widetilde{b}+fp_1)^*}{h\phi_1} \in H(\mathbb{C}), \\ (a - vq)^* & = & \left(\frac{\widetilde{a}}{h\psi_1} - \frac{f}{h\phi_1\psi_1}q_1\phi_1\right)^* = \frac{(\widetilde{a}-fq_1)^*}{h\psi_1} \in H(\mathbb{C}). \end{array}\right\} \tag{3.7}$$

Hence we have to look for a polynomial $f \in \mathbb{R}[s]$ which places the zeros of $\widetilde{b}^* + fp_1^*$ and $\widetilde{a}^* - fq_1^*$ appropriately at the same time. In the rest of the proof we will show that these are two interpolation problems for f, which can in fact be solved both with the same polynomial $f \in \mathbb{R}[s]$.

Firstly, for $\alpha \in \mathcal{V}(\phi_1 h)$ one has $p_1^*(\alpha) \neq 0$, since:
i) if $\alpha \in \mathcal{V}(\phi_1) \subset \mathcal{V}(q^*)$, then $\alpha \notin \mathcal{V}(p^*)$, hence $\alpha \notin \mathcal{V}(p_1^*)$.
ii) If $h(\alpha) = 0$, then by (3.6) and (3.4) it follows $0 = \widetilde{a}^*(\alpha)p_1^*(\alpha) + \widetilde{b}^*(\alpha)q_1^*(\alpha)$ and $\widetilde{a}^*(\alpha) \neq 0 \neq \widetilde{b}^*(\alpha)$. Therefore $\mathcal{V}(p^*, q^*) = \emptyset$ yields $p_1^*(\alpha) \neq 0 \neq q_1^*(\alpha)$.

From this we obtain for $\alpha \in \mathcal{V}(\phi_1 h)$:

$$\begin{aligned}
\mu_\alpha(\widetilde{b}^* + fp_1^*) \geq k \iff & (\widetilde{b}^* + fp_1^*)^{(\nu)}(\alpha) = 0,\ \nu = 0, \ldots, k-1 \\
\iff & \widetilde{b}^{*(\nu)}(\alpha) + \sum_{\mu=0}^{\nu} \tbinom{\nu}{\mu} f^{(\mu)}(\alpha) p_1^{*(\nu-\mu)}(\alpha) = 0,\ \nu = 0, \ldots, k-1 \\
\iff & f^{(\nu)}(\alpha) = -\tfrac{1}{p_1^*(\alpha)} \left[\widetilde{b}^{*(\nu)}(\alpha) + \textstyle\sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} p_1^{*(\nu-\mu)}(\alpha) f^{(\mu)}(\alpha)\right] \\
& \text{for } \nu = 0, \ldots, k-1.
\end{aligned}$$

A similar result holds for $\alpha \in \mathcal{V}(\psi_1 h)$.

As a consequence $f \in \mathbb{R}[s]$ satisfies (3.7) if and only if

$$f^{(\nu)}(\alpha) = \begin{cases} -\frac{1}{p_1^*(\alpha)} \left[\widetilde{b}^{*(\nu)}(\alpha) + \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} p_1^{*(\nu-\mu)}(\alpha) f^{(\mu)}(\alpha)\right] & \text{if } \alpha \in \mathcal{V}(\phi_1 h) \\ \text{for } \nu = 0, \ldots, \mu_\alpha(\phi_1 h) - 1 & \\ \frac{1}{q_1^*(\alpha)} \left[\widetilde{a}^{*(\nu)}(\alpha) - \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} q_1^{*(\nu-\mu)}(\alpha) f^{(\mu)}(\alpha)\right] & \text{if } \alpha \in \mathcal{V}(\psi_1 h) \\ \text{for } \nu = 0, \ldots, \mu_\alpha(\psi_1 h) - 1 & \end{cases} \tag{3.8}$$

In particular, for $\alpha \in \mathcal{V}(\phi_1 h) \cap \mathcal{V}(\psi_1 h) = \mathcal{V}(h)$ and $\nu = 0 \ldots, \mu_\alpha(h) - 1$ the derivative $f^{(\nu)}(\alpha)$ has to be equal to both expressions given in (3.8). Thus we can find such an f only if for those α and ν it is true that

$$-\frac{1}{p_1^*(\alpha)} \left[\widetilde{b}^{*(\nu)}(\alpha) + \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} p_1^{*(\nu-\mu)}(\alpha) f^{(\mu)}(\alpha)\right] = \frac{1}{q_1^*(\alpha)} \left[\widetilde{a}^{*(\nu)}(\alpha) - \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} q_1^{*(\nu-\mu)}(\alpha) f^{(\mu)}(\alpha)\right].$$

But this is indeed valid, since from (3.6) it follows

$$0 = h^{(\nu)}(\alpha) = (\widetilde{a}^* p_1^* + \widetilde{b}^* q_1^*)^{(\nu)}(\alpha) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \widetilde{a}^{*(\mu)}(\alpha) p_1^{*(\nu-\mu)}(\alpha) + \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \widetilde{b}^{*(\mu)}(\alpha) q_1^{*(\nu-\mu)}(\alpha)$$

for $\nu = 0, \ldots, \mu_\alpha(h) - 1$ and therefore one can apply Lemma A.2.

Since $\mathcal{V}(\phi_1 \psi_1 h) \subseteq \mathbb{C}$ is symmetric with respect to complex conjugation, Prop. 3.1 a) and Lemma A.1 imply the existence of $f \in \mathbb{R}[s]$ with the properties required in (3.8). $\square$

Example 3.3 Let $p = s^2$, $q = z - 1 \in \mathcal{H}$. Then $s \,|\, q^*$ but $s^2 \not|\, q^*$ in $H(\mathbb{C})$, thus $d = s$ is a gcd of p, q. A Bézout-equation is given by

$$s = \frac{(1-s)z + 2s - 1}{s^2} s^2 + (s-1)(z-1).$$

Note also that $\ker \widetilde{p} = \{w \in \mathcal{C}^\infty(\mathbb{R}) | \exists \alpha, \beta \in \mathbb{R}\, \forall t \in \mathbb{R} : w(t) = \alpha + \beta t\}$ and $\ker \widetilde{q} = \{w \in \mathcal{C}^\infty(\mathbb{R}) | w \text{ is of period } 1\}$, hence $\ker \widetilde{p} \cap \ker \widetilde{q} = \{w \in \mathcal{C}^\infty(\mathbb{R}) | w \text{ constant }\} = \ker \widetilde{d}$.

It is a standing conjecture, that every commutative Bézout-domain is an *elementary divisor domain*, which means by definition, that matrices can be brought into diagonal form via left-right equivalence, see e. g. [3, p. 92]. In the present case, one can in fact prove the elementary divisor property. To do so, we will show the following lemma, which states that $\mathcal{H}$ is a so-called *adequate ring*, see e. g. [12, p. 473].

Lemma 3.4 *Let $p, q \in \mathcal{H}$, $p \neq 0$. There exists a factorization $p = ab$ with $a, b \in \mathcal{H}$ such that a and q are coprime whereas $\hat{b}$ and q are not coprime whenever $\hat{b} \in \mathcal{H}\backslash\mathcal{H}^\times$ is a divisor of b.*

PROOF: The idea of the proof is as follows: factorize $p = ab$ such that $\mathcal{V}(b^*) = \mathcal{V}(p^*, q^*)$ and $\mu_\lambda(b^*) = \mu_\lambda(p^*)$ for all $\lambda \in \mathcal{V}(b^*)$. This can easily be done if $\#\mathcal{V}(p^*, q^*) < \infty$. In the infinite case it needs an iterative procedure as described below.

Let $b_1 \in \mathcal{H}$ be a gcd of p and q and put $a_1 = \frac{p}{b_1}$, so that $p = a_1 b_1$. Define successively for $p = a_i b_i$, $i \in \mathbb{N}$, the following elements:

$$\text{let } c_i \in \mathcal{H} \text{ be a gcd of } a_i \text{ and } b_i;\ \text{put } a_{i+1} = \frac{a_i}{c_i} \text{ and } b_{i+1} = c_i b_i. \tag{3.9}$$

Hence $p = a_i b_i = a_{i+1} c_i b_i = a_{i+1} b_{i+1}$. This gives a sequence of elements $a_i \in \mathcal{H}$ with the property that a_{i+1} divides a_i in $\mathcal{H}$. But then a_{i+1} divides a_i also in the principal ideal ring $\mathbb{R}(s)[z, z^{-1}]$ with the consequence that for some $k \in \mathbb{N}$ there exist $l \in \mathbb{Z}$ and $\phi \in \mathbb{R}[s]\backslash\{0\}$ such that $c_k = \phi z^l$ is a unit in $\mathbb{R}(s)[z, z^{-1}]$. Thus the procedure (3.9) yields the existence of a factorization

$$p = a_k b_k \text{ with } \phi \in \mathbb{R}[s] \text{ as a gcd of } a_k \text{ and } b_k \text{ in } \mathcal{H}.$$

This implies that $\mathcal{V}(a_k^*, b_k^*)$ is finite, say $\mathcal{V}(a_k^*, b_k^*) = \{\lambda_1, \ldots, \lambda_n\}$ and we can define $f := \prod_{i=1}^n (s - \lambda_i)^{l_i} \in \mathbb{R}[s]$ where $l_i = \mu_{\lambda_i}(a_k^*)$. With $a := a_k f^{-1} \in \mathcal{H}$ and $b := f b_k \in \mathcal{H}$ we get the factorization $p = ab$, which in fact satisfies the requirements of the lemma:
1) To establish the coprimeness of a and q, suppose $\mathcal{V}(a^*, q^*) \neq \emptyset$. Thus let $\lambda \in \mathcal{V}(a^*, q^*) \subseteq \mathcal{V}(p^*, q^*) = \mathcal{V}(b_1^*)$. Then $\lambda \in \mathcal{V}(b_1^*, a_k^*) \subseteq \mathcal{V}(a_k^*, b_k^*) = \{\lambda_1, \ldots, \lambda_n\}$. But for $\lambda = \lambda_j$ it is $\mu_\lambda(a^*) = \mu_{\lambda_j}(a_k^*) - \mu_{\lambda_j}(f) = 0$. Hence $\mathcal{V}(a^*, q^*) = \emptyset$ and from Prop. 3.1 f) we conclude the coprimeness of a and q.
2) Let $\hat{b} \in \mathcal{H}\backslash\mathcal{H}^\times$ be a divisor of b and fix some $\lambda \in \mathcal{V}(b^*)$ with $\hat{b}^*(\lambda) = 0$. The construction (3.9) of the sequences (c_i) and (b_i) leads to the following equality of zero sets (note that we count zeros in $\mathcal{V}$ not with multiplicity)

$$\mathcal{V}(b^*) = \mathcal{V}(f^* b_k^*) = \mathcal{V}(b_k^*) = \mathcal{V}(c_{k-1}^* b_{k-1}^*) = \mathcal{V}(b_{k-1}^*) = \ldots = \mathcal{V}(b_1^*) = \mathcal{V}(p^*, q^*).$$

Thus $\lambda \in \mathcal{V}(q^*, \hat{b}^*)$ and therefore $\hat{b}$ and q are not coprime.

Note that in the case $\mathcal{V}(p^*, q^*) = \{\lambda_1, \ldots, \lambda_n\}$ is finite, the above construction leads to the factorization $p = \frac{p}{b} b$ with $b = \prod_{i=1}^n (s - \lambda_i)^{l_i}$ and $l_i = \mu_{\lambda_i}(p^*)$. □

Now we can summarize the properties for matrices over $\mathcal{H}$, as they follow from the above ring theoretic results.

Theorem 3.5

a) *Let* $a_1, \ldots, a_n \in \mathcal{H}$ *and* $d \in \mathcal{H}$ *be a gcd of* $a_1, \ldots, a_n$. *Then there exists a matrix* $A \in \mathcal{H}^{n\times n}$ *with* $[a_1, \ldots, a_n]$ *as its first row and* $\det A = d$.

b) *For* $P \in \mathcal{H}^{n\times m}$ *there exists* $U \in Gl_n(\mathcal{H})$ *so that* $UP \in \mathcal{H}^{n\times m}$ *has upper triangular form.*

c) *Let* $P \in \mathcal{H}^{n\times m}$ *and* $Q \in \mathcal{H}^{l\times m}$. *There exists a greatest common right divisor (gcrd)* $D \in \mathcal{H}^{m\times m}$ *and matrices* $A \in \mathcal{H}^{m\times n}$, $B \in \mathcal{H}^{m\times l}$ *with* $D = AP + BQ$. *If* $\mathrm{rk}D = m$, *then* D *is unique modulo multiplication from the left by unimodular matrices.*

d) *Let* $P, Q \in \mathcal{H}^{m\times m}$ *with* $\mathrm{rk}P = \mathrm{rk}Q = m$. *Then there exists a least common left multiple (lclm)* $M \in \mathcal{H}^{m\times m}$, *which is unique modulo unimodular factors from the left.*

e) $\mathcal{H}$ *is an elementary divisor ring, that is, for* $P \in \mathcal{H}^{n\times m}$ *with* $\mathrm{rk}P = r$ *there exist* $U \in Gl_n(\mathcal{H})$ *and* $V \in Gl_m(\mathcal{H})$ *such that*

$$UPV = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{H}^{n\times m} \quad \text{with } P_1 = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p_r \end{bmatrix} \in \mathcal{H}^{r\times r} \tag{3.10}$$

where $p_i \neq 0$ *for all* i *and* $p_i \,|\, p_{i+1}$ *for* $i = 1, \ldots, r-1$.

Proof: The parts a) – d) hold in general for matrices over commutative Bézout-domains. The proof of these parts is identical with that given for principal ideal domains in [13, p. 31–36]. Part e) follows from Lemma 3.4, as shown in [12, p. 473] for arbitrary adequate rings. □

The existence of a lcm for elements $p, q \in \mathcal{H}$ as we proved in Prop. 3.1 g) can also be concluded from part a) of the above theorem (see e. g. [4, p. 126, Cor. 2]).

4 Correspondence between behaviors and ideals in $\mathcal{H}$

The results in section 3 enable us to show a correspondence between the lattice of behaviors associated with delay-differential equations of the type (1.1) and the lattice of finitely generated ideals in $\mathcal{H}$. After introducing multivariable delay-differential operators, an analogous version of this correspondence will be shown also in that case.

Remember that, as outlined in Def. 2.7 and Remark 2.8, for $p \in \mathcal{H}$ the operator $\widetilde{p} \in \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}))$ exists. In particular, for $p \in \mathbb{R}[s, z] \subset \mathcal{H}$ this includes the classical case as in equation (1.1).

Proposition 4.1 *For* $p, q \in \mathcal{H}\backslash\{0\}$ *let* $d \in \mathcal{H}$ *be a gcd of* p, q *and* $c \in \mathcal{H}$ *be a lcm of* p, q. *Then*

a) $\ker \widetilde{p} \subseteq \ker \widetilde{q} \Longleftrightarrow p \,|\, q$,

b) $\ker \widetilde{d} = \ker \widetilde{p} \cap \ker \widetilde{q}$,

c) $\ker \widetilde{c} = \ker \widetilde{p} + \ker \widetilde{q}$, *d) If $d \in \mathcal{H}^\times$, then $\ker \widetilde{p} + \ker \widetilde{q} = \ker \widetilde{pq} = \ker \widetilde{qp}$,*

e) Let $a \in \mathcal{H}$ be such that $\ker \widetilde{p} \cap \ker \widetilde{q} \subseteq \ker \widetilde{a}$. Then $a \in p\mathcal{H} + q\mathcal{H}$.

PROOF: a) "$\Rightarrow$" Let $p = a\phi^{-1}$, $q = b\phi^{-1}$ with $a, b \in \mathcal{R}$ and $\phi \in \mathbb{R}[s]$. Then it is easy to see that $\ker \widetilde{p} \subseteq \ker \widetilde{q}$ implies $\ker \widetilde{a} \subseteq \ker \widetilde{b}$. Thus by (2.2) one has $b^*(a^*)^{-1} = q^*(p^*)^{-1} \in H(\mathbb{C})$ and with Prop. 3.1 e) it follows $p \,|\, q$ in $\mathcal{H}$.
"$\Leftarrow$" If $q = ap$ with some $a \in \mathcal{H}$, then part 3) of Remark 2.8 yields $\widetilde{q} = \widetilde{a} \circ \widetilde{p}$ and therefore $\ker \widetilde{p} \subseteq \ker \widetilde{q}$.

b) is a consequence of a) and the existence of a Bézout-equation $d = ap + bq$ in $\mathcal{H}$ together with part 3) of Remark 2.8.

c) "$\supseteq$" follows from a).
"$\subseteq$" Let $p = ad$, $q = bd$ with a, $b \in \mathcal{H}$. Then, by Prop. 3.1 g) we can take $c = abd$ as a lcm of p, q. By coprimeness of a, b there exists f, $g \in \mathcal{H}$ with $1 = af + bg$. Hence $w \in \ker \widetilde{c}$ satisfies $w = \widetilde{fa}(w) + \widetilde{gb}(w) \in \ker \widetilde{q} + \ker \widetilde{p}$.

d) If $d \in \mathcal{H}^\times$, then pq is a lcm of p, q, hence the claim holds by c).

e) follows from a) and b) and the equality $d\mathcal{H} = p\mathcal{H} + q\mathcal{H}$. □

Notice that the examples 2.9 and 3.3 correspond to the situation given in a) and b) of the above proposition.

Now we will come to the multivariable case. From Remark 2.8 we conclude that for a matrix $P = (p_{ij}) \in \mathcal{H}^{n\times m}$ the operator

$$\begin{array}{rccl} \widetilde{P}: & \mathcal{C}^\infty(\mathbb{R}^m) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^n) \\ & (w_1, \ldots, w_m)^{\mathsf{t}} & \longmapsto & \Big(\sum_{j=1}^{m} \widetilde{p_{1j}}(w_j), \ldots, \sum_{j=1}^{m} \widetilde{p_{nj}}(w_j)\Big)^{\mathsf{t}} \end{array}$$

is well-defined. Thus the behavior, defined by a system of delay-differential equations, can be described as $\ker \widetilde{P}$ with some $P \in \mathcal{R}^{n\times m}$ or in the more general case $P \in \mathcal{H}^{n\times m}$.

Remark 4.2

a) The map $P \mapsto \widetilde{P}$ from $\mathcal{H}^{n\times m}$ to $\mathrm{Hom}_{\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}^m), \mathcal{C}^\infty(\mathbb{R}^n))$ is $\mathbb{R}$-linear, injective and satisfies $\widetilde{PQ} = \widetilde{P} \circ \widetilde{Q}$ for $P \in \mathcal{H}^{n\times m}$, $Q \in \mathcal{H}^{m\times l}$.

b) Analogously to the scalar case in Def. 2.4 a) the map

$$\begin{array}{rcl} \mathcal{H}^{n\times m} & \longrightarrow & H(\mathbb{C})^{n\times m} \\ P & \longmapsto & P^*(s) := P(s, e^{-s}) \end{array}$$

is a well-defined $\mathbb{R}$-linear map and satisfies $(PQ)^*(s) = P^*(s)Q^*(s)$ for $P \in \mathcal{H}^{n\times m}$, $Q \in \mathcal{H}^{m\times l}$.

Let us first list some properties of the operator $\widetilde{P}$.

Proposition 4.3 *Let $P \in \mathcal{H}^{n\times m}$. Then*

a) If $n = m$ and $P \in Gl_n(\mathcal{H})$, then $\widetilde{P}$ is bijective and $P^*(s) \in Gl_n(\mathbb{C})$ for all $s \in \mathbb{C}$.

b) $\widetilde{P}$ is surjective if and only if $\mathrm{rk}P = n$.

c) The following properties are equivalent: i) $\widetilde{P}$ is injective, ii) $\mathrm{rk}P^*(s) = m$ *for all $s \in \mathbb{C}$, iii) there exists $Q \in \mathcal{H}^{m\times n}$ with $QP = I_m$.*

PROOF: a) follows from the existence of $Q \in \mathcal{H}^{n\times n}$ with $PQ = QP = I_n$ together with Remark 4.2.

b) Let $\mathrm{rk}P = r \leq n$. By Thm. 3.5 e) there exist $U \in Gl_n(\mathcal{H})$ and $V \in Gl_m(\mathcal{H})$ so that UPV is as in (3.10). By a) $\widetilde{P}$ is surjective iff $\widetilde{UPV}$ is surjective and together with Prop. 2.10 this holds iff $r = n$.

c) All three properties are invariant under multiplication with unimodular matrices from the left or from the right. Thus, using again Thm. 3.5 e), we can restrict ourselves to diagonal P. Since all three properties imply $\mathrm{rk}P = m$, we can assume

$$P = \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \in \mathcal{H}^{n\times m} \text{ with } P_1 = \mathrm{diag}(p_1, \ldots, p_m) \in \mathcal{H}^{m\times m}.$$

Now i) implies the injectivity of $\widetilde{p_i}$, thus, with (2.2) and Prop. 3.1 b), $p_i \in \mathcal{H}^\times$. This yields ii). In the same way, ii) leads to $p_i \in \mathcal{H}^\times$ for all i and iii) can be concluded. The implication "iii) $\Rightarrow$ i)" follows from Remark 4.2 a). □

Now we can generalize part of the results in Prop. 4.1 to the multivariable case.

Proposition 4.4 *Let $P \in \mathcal{H}^{n\times m}$, $Q \in \mathcal{H}^{l\times m}$ and $D \in \mathcal{H}^{m\times m}$ be a gcrd of P, Q. Then*

a) $\ker \widetilde{P} \cap \ker \widetilde{Q} = \ker \widetilde{D}$,

b) P is a right divisor of Q if and only if $\ker \widetilde{P} \subseteq \ker \widetilde{Q}$,

c) under the condition $\mathrm{rk}P = n$, $\mathrm{rk}Q = l$ *the following holds true:* $\ker \widetilde{P} = \ker \widetilde{Q}$ *iff $n = l$ and $P = UQ$ for some $U \in Gl_n(\mathcal{H})$.*

PROOF: a) Since "$\Rightarrow$" of b) holds by Remark 4.2 a), part a) follows from the existence of a Bézout-equation for D (see Thm. 3.5 c)).

b) It remains to prove "$\Leftarrow$".
Let $r = \mathrm{rk}P$ and $U \in Gl_n(\mathcal{H})$, $V \in Gl_m(\mathcal{H})$ be such that $P' = UPV$ is as in (3.10). Denoting $Q' = UQV$, Prop. 4.3 a) implies $\ker \widetilde{P'} \subseteq \ker \widetilde{Q'}$. This yields $Q' = [R, 0]$ with $R \in \mathcal{H}^{l\times r}$ and moreover, $\ker \widetilde{p_j} \subseteq \ker \widetilde{R_{ij}}$ for all $j = 1, \ldots, r$ and $i = 1, \ldots, l$. Hence, using Prop. 4.1 a) we get the existence of $A \in \mathcal{H}^{l\times n}$ such that $AP' = Q'$ and therefore $U^{-1}AUP = Q$.

c) "$\Leftarrow$" is obvious.
"$\Rightarrow$" By b) there exist $P = UQ$ and $Q = VP$ for some $U \in \mathcal{H}^{n\times l}$, $V \in \mathcal{H}^{l\times n}$. Then the full rank assumption implies $VU = I_l$ and $UV = I_n$ which leads to the desired result. □

5 Controllability

In this section we will generalize the well-known Hautus-criterion for controllability to delay-differential systems. For time-delay state-space-systems this criterion characterizes spectral controllability, as it is known from e. g. [18] and [2]. In the behavioral context this criterion is established for finite-dimensional discrete- or continuous-time AR-systems (see e. g. [22, Prop. 4.3]) and, very recently, in [19] for exactly the same situation of delay-differential equations as presented in the paper at hand. However, the proof in [19] uses quite different methods than those developed in this paper.

Whereas controllability for state-space-systems is formulated, of course, in terms of control functions and state trajectories, we do not have this possibility for behaviors. Hence we will use the notion of controllability as defined in [22]. For this we have to introduce first the concatenation of two functions.

Definition 5.1 *Let* $-\infty \leq a_1 < a_2 \leq b_1 < b_2 \leq \infty$ *and* $w_1 : (a_1, b_1) \rightarrow \mathbb{R}^m$ *and* $w_2 : [a_2, b_2) \rightarrow \mathbb{R}^m$ *be two functions. For* $t_0 \in [a_2, b_1]$ *denote by* $w_1 \wedge_{t_0} w_2 : (a_1, b_2) \rightarrow \mathbb{R}^m$ *the following concatenation of* w_1 *and* w_2 *at* t_0

$$(w_1 \wedge_{t_0} w_2)(t) := \begin{cases} w_1(t) & \text{for } a_1 < t < t_0 \\ w_2(t) & \text{for } t_0 \leq t < b_2 \end{cases}$$

Using this definition, a behavior is called controllable if it is closed under concatenation in the sense given below. In [22, pp. 186] one can find the system theoretic justification of this notion.

Definition 5.2 *Let* $\mathcal{B}$ *be a shift-invariant subspace of* $\mathcal{C}^\infty(\mathbb{R}^m)$*. Then* $\mathcal{B}$ *is called controllable, if it satisfies: for all* $w,\, w' \in \mathcal{B}$ *there exists* $t_0 \geq 0$ *and* $c \in \mathcal{C}^\infty([0, t_0), \mathbb{R}^m)$ *with* $w \wedge_0 c \wedge_{t_0} \sigma^{t_0} w' \in \mathcal{B}$*.*

The requirement $w \wedge_0 c \wedge_{t_0} \sigma^{t_0} w' \in \mathcal{B}$ yields in particular, that the concatenation is in $\mathcal{C}^\infty(\mathbb{R}^m)$.

Note that $\mathcal{C}^\infty(\mathbb{R}^m)$ is controllable, more strongly $\mathcal{C}^\infty(\mathbb{R}^m)$ is controllable in arbitrary short time: for all $w,\, w' \in \mathcal{C}^\infty(\mathbb{R}^m)$ and all $t_0 > 0$ there exists $c \in \mathcal{C}^\infty([0, t_0), \mathbb{R}^m)$ with $w \wedge_0 c \wedge_{t_0} \sigma^{t_0} w' \in \mathcal{C}^\infty(\mathbb{R}^m)$.

Since we introduce the concept of controllability only for shift-invariant subspaces, it makes sense to consider only controllability at time zero.

Whereas it is obvious that for $U \in \mathbb{R}[s]^{n \times m}$ and $w,\, w' \in \mathcal{C}^\infty(\mathbb{R}^m)$ it is $\widetilde{U}(w \wedge_0 w') = \widetilde{U}(w) \wedge_0 \widetilde{U}(w')$ if $w \wedge_0 w'$ is sufficiently differentiable at $t_0 = 0$, it is a priori not clear, that $\widetilde{U}(w \wedge_0 w')$ is a sort of concatenation of $\widetilde{U}(w)$ and $\widetilde{U}(w')$ if $U \in \mathbb{R}[s,z]^{n \times m}$ or even $U \in \mathcal{H}^{n \times m}$.

Lemma 5.3 *Let* $U = \sum_{j=0}^{L} U_j z^j \in \mathbb{R}[s,z]^{n \times m}$ *with* $U_j \in \mathbb{R}[s]^{n \times m}$*. Further let* $w,\, w' \in \mathcal{C}^\infty(\mathbb{R}^m)$*,* $t_0 \in \mathbb{R}$ *with* $w \wedge_{t_0} w' \in \mathcal{C}^\infty(\mathbb{R}^m)$*. Then there exists* $c \in \mathcal{C}^\infty([t_0, t_0 + L), \mathbb{R}^n)$ *so that* $\widetilde{U}(w \wedge_{t_0} w') = \widetilde{U}(w) \wedge_{t_0} c \wedge_{t_0 + L} \widetilde{U}(w')$*.*

PROOF: A direct calculation shows

$$\begin{aligned}
\widetilde{U}(w\wedge_{t_0}w')(t) &= \sum_{j=0}^{L}\widetilde{U}_j(w\wedge_{t_0}w')(t-j) = \sum_{j=0}^{L}(\widetilde{U}_j(w)\wedge_{t_0}\widetilde{U}_j(w'))(t-j) \\
&= \begin{cases} \sum_{j=0}^{L}\widetilde{U}_j(w')(t-j) = \widetilde{U}(w')(t) & \text{if } t \geq t_0+L \\ c(t) & \text{if } t_0 \leq t < t_0+L \\ \sum_{j=0}^{L}\widetilde{U}_j(w)(t-j) = \widetilde{U}(w)(t) & \text{if } t < t_0 \end{cases}
\end{aligned}$$

for some function $c : [t_0, t_0+L) \to \mathbb{R}^n$. Hence $\widetilde{U}(w\wedge_{t_0}w') = \widetilde{U}(w)\wedge_{t_0}c\wedge_{t_0+L}\widetilde{U}(w')$. Since $\widetilde{U}(w\wedge_{t_0}w') \in \mathcal{C}^\infty(\mathbb{R}^n)$, we also get $c \in \mathcal{C}^\infty([t_0, t_0+L), \mathbb{R}^n)$. □

With this knowledge we can prove

Lemma 5.4 *Let $\mathcal{B}$ be a shift-invariant linear controllable subspace of $\mathcal{C}^\infty(\mathbb{R}^m)$ and let $U \in \mathcal{H}^{n\times m}$. Then $\widetilde{U}(\mathcal{B})$ is a shift-invariant linear controllable subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$.*

PROOF: Since $\mathcal{B}$ is shift-invariance, it is enough to consider $U = \sum_{j=0}^{L} U_j z^j \in \mathbb{R}(s)[z]^{n\times m}$ with $U_j \in \mathbb{R}(s)^{n\times m}$.

Let $w, w' \in \mathcal{B}$. Then $\sigma^L w' \in \mathcal{B}$ and there exist $t_0 \geq 0$ and $c \in \mathcal{C}^\infty([0,t_0), \mathbb{R}^m)$ so that $\bar{w} := w\wedge_0 c\wedge_{t_0}\sigma^{t_0+L}w' \in \mathcal{B}$.

<u>1. case:</u> Let $U_j \in \mathbb{R}[s]^{n\times m}$ for all j, thus $U \in \mathbb{R}[s,z]^{n\times m}$. Then by Lemma 5.3 we get the existence of $c' \in \mathcal{C}^\infty([0, t_0+L), \mathbb{R}^n)$ so that $\widetilde{U}(\bar{w}) = \widetilde{U}(w\wedge_0 c\wedge_{t_0}\sigma^{t_0+L}w') = \widetilde{U}(w)\wedge_0 c'\wedge_{t_0+L}\widetilde{U}(\sigma^{t_0+L}w') = \widetilde{U}(w)\wedge_0 c'\wedge_{t_0+L}\sigma^{t_0+L}\widetilde{U}(w') \in \widetilde{U}(\mathcal{B})$. Since $w, w' \in \mathcal{B}$ were arbitrary, this yields the controllability of $\widetilde{U}(\mathcal{B})$.

<u>2. case:</u> Let $U_j = V_j\phi^{-1}$ with $V_j \in \mathbb{R}[s]^{n\times m}$. Put $V = \sum_{j=0}^{L} V_j z^j \in \mathbb{R}[s,z]^{n\times m}$. Then $U = V\phi^{-1}$ and by definition $\widetilde{U}(\bar{w}) = \widetilde{V}(v)$, if $v \in \mathcal{C}^\infty(\mathbb{R}^m)$ fulfills $\widetilde{\phi}(v) = \bar{w}$.

As in the first case, we shall show that $\widetilde{U}(\bar{w})$ is a concatenation of $\widetilde{U}(w)$ and $\sigma^{t_0+L}\widetilde{U}(w')$, so that $\widetilde{U}(\bar{w}) \in \widetilde{U}(\mathcal{B})$ implies the controllability of $\widetilde{U}(\mathcal{B})$. In order to do so, we will construct a solution of $\widetilde{\phi}(v) = \bar{w}$ which corresponds to the special form of $\bar{w} = w\wedge_0 c\wedge_{t_0}\sigma^{t_0+L}w'$. For this let $c' \in \mathcal{C}^\infty([0,t_0), \mathbb{R}^m)$ be so that $\widetilde{\phi}(c') = c$. Then the solutions $v_i \in \mathcal{C}^\infty(\mathbb{R}^m)$, $i = 1, 2$ of

$$\begin{aligned}
&\widetilde{\phi}(v_1) = w, && v_1^{(\nu)}(0) = c'^{(\nu)}(0) \text{ for } \nu = 0, \dots, \deg\phi - 1 \\
&\widetilde{\phi}(v_2) = \sigma^{t_0+L}w', && v_2^{(\nu)}(t_0) = c'^{(\nu)}(t_0) \text{ for } \nu = 0, \dots, \deg\phi - 1
\end{aligned}$$

satisfy $v := v_1\wedge_0 c'\wedge_{t_0}v_2 \in \mathcal{C}^\infty(\mathbb{R}^m)$ and $\widetilde{\phi}(v) = \bar{w}$. Moreover, $\widetilde{V}(v_1) = \widetilde{U}(w)$, $\widetilde{V}(v_2) = \widetilde{U}(\sigma^{t_0+L}w')$. Now, by the first case of this proof there exists $c'' \in \mathcal{C}^\infty([0, t_0+L), \mathbb{R}^n)$ so that

$$\begin{aligned}
\widetilde{U}(\bar{w}) &= \widetilde{V}(v) = \widetilde{V}(v_1\wedge_0 c'\wedge_{t_0}v_2) = \widetilde{V}(v_1)\wedge_0 c''\wedge_{t_0+L}\widetilde{V}(v_2) = \widetilde{U}(w)\wedge_0 c''\wedge_{t_0+L}\widetilde{U}(\sigma^{t_0+L}w') \\
&= \widetilde{U}(w)\wedge_0 c''\wedge_{t_0+L}\sigma^{t_0+L}\widetilde{U}(w') \in \widetilde{U}(\mathcal{B}).
\end{aligned}$$

□

Now we can prove the main part of this section

Theorem 5.5 *Let $P \in \mathcal{H}^{n\times m}$. Then $\ker \widetilde{P}$ is controllable if and only if $\text{rk}P^*(s) = \text{rk}P$ for all $s \in \mathbb{C}$.*

Proof: a) We first prove the scalar case $p \in \mathcal{H}$. If $p = 0$ then obviously $\ker \widetilde{p} = \mathcal{C}^\infty(\mathbb{R})$ is controllable. Let $p \neq 0$.
"$\Leftarrow$" holds, since $\ker \widetilde{p} = \{0\}$ if $p \in \mathcal{H}^\times$.
"$\Rightarrow$" Let $w_1 \in \ker \widetilde{p}$. Then there exist $t_0 > 0$ and some $c \in \mathcal{C}^\infty([0, t_0), \mathbb{R})$ with $v := w_1 \wedge_0 c \wedge_{t_0} 0 \in \ker \widetilde{p}$ and Prop. 2.10 b) implies $v = 0$, hence, again by Prop. 2.10 b), $w_1 = 0$. Therefore controllability of $\ker \widetilde{p}$ implies $\ker \widetilde{p} = \{0\}$ and from Prop. 3.1 b) it follows $p \in \mathcal{H}^\times$.

b)Let $P \in \mathcal{H}^{n\times m}$. Using Thm. 3.5 e) and Lemma 5.4 we can restrict ourselves to the case of P being as in (3.10).
"$\Leftarrow$" The assumption on the rank implies that $p_j \in \mathcal{H}^\times$ for $j = 1, \ldots, r$ and therefore $\ker \widetilde{P} = \{(0, \ldots, 0, w_{r+1}, \ldots, w_m)^{\mathsf{t}} \,|\, w_i \in \mathcal{C}^\infty(\mathbb{R}),\ i = r+1, \ldots, m\}$ which is indeed controllable.
"$\Rightarrow$" The controllability of $\ker \widetilde{P}$ yields the controllability of $\ker \widetilde{p_j}$ for $j = 1, \ldots, r$. Hence by the scalar case $p_j \in \mathcal{H}^\times$ and the desired conclusion follows. □

Conclusions.
As can be seen from section 4) and 5), the ring $\mathcal{H}$ seems to be the adequate object for an algebraic treatment of delay-differential equations as (1.1) and (1.2). Once the algebraic properties of $\mathcal{H}$ are established, the translation into properties of the solution spaces are nearly straightforward.

In a forthcoming paper it will be shown, how the existence of image-representations for the systems under investigation can be characterized with the help of this algebraic framework. Moreover, the analytical meaning of the operators in $\mathcal{H}$ has to be clearified.

A Appendix

Proof of Prop. 2.10:
a) Let $p \in \mathcal{H}\backslash\{0\}$ and $v \in \mathcal{C}^\infty(\mathbb{R})$. We have to find $w \in \mathcal{C}^\infty(\mathbb{R})$ fulfilling $\widetilde{p}(w) = v$.

First, it suffices to assume $p \in \mathcal{R}$, for let $p = q\phi^{-1}$ with $q \in \mathcal{R}$, $\phi \in \mathbb{R}[s]$. If we find $f \in \mathcal{C}^\infty(\mathbb{R})$ with $\widetilde{q}(f) = v$ and put $\widetilde{\phi}(f) = w$, then we have $\widetilde{p}(w) = v$. Hence we need to show the surjectivity of $\widetilde{q}$.

Thus let $p \in \mathcal{R}$ and, more precisely,

$$p = \sum_{j=0}^{L} p_j z^j \in \mathbb{R}[s, z] \text{ with } p_j \in \mathbb{R}[s] \text{ and } L \geq 1.$$

Put $p_0 = \sum_{i=0}^{l} a_i s^i$, $a_l = 1$ and $p_L = \sum_{i=0}^{r} b_i s^i$, $b_r \neq 0$.

We will construct piecewise a function $w \in \mathcal{C}^\infty(\mathbb{R})$ which fulfills for all $t \in \mathbb{R}$

$$\widetilde{p}(w)(t) = \sum_{j=0}^{L} \widetilde{p_j}(w)(t-j) = v(t). \tag{A.1}$$

The idea of the construction is as follows: Start with a function $w_0 \in \mathcal{C}^\infty[0,L]$. In order to extend w_0 via concatenation (see Def. 5.1) to a solution of $\widetilde{p}(w) = v$ one has to solve successively ordinary inhomogeneous differential equations of the type

$$\begin{aligned} \widetilde{p_0}(\bar{w}_{k+1}) &= v - \widetilde{(p-p_0)}(w_k) \text{ on the time interval } [L+k, L+k+1] \text{ for } k \geq 0, \\ \widetilde{p_L}(\bar{w}_k) &= \sigma^{-L}\Big(v - \widetilde{(p-p_L)}(w_{k+1})\Big) \text{ on the time interval } [k, k+1] \text{ for } k \leq -1, \end{aligned}$$

where the right hand sides are determined successively by

$$\begin{aligned} w_k &= w_0 \wedge_L \bar{w}_1 \wedge_{L+1} \dots \wedge_{L+k-1} \bar{w}_k \text{ on } [0, L+k] \text{ for } k \geq 1 \\ w_{k+1} &= \bar{w}_{k+1} \wedge_{k+2} \dots \wedge_{-1} \bar{w}_{-1} \wedge_0 w_0 \text{ on } [k+1, L] \text{ for } k < -1. \end{aligned}$$

The initial conditions at the points $L+k$ (for $k \geq 0$) and $k+1$ (for $k \leq -1$) have, of course, to be prescribed such that the concatenations are as smooth as possible. If one chooses the initial function $w_0 \in \mathcal{C}^\infty[0,L]$ appropriately, this procedure leads indeed to infinitely smooth concatenations and thus to a solution $w \in \mathcal{C}^\infty(\mathbb{R})$ of $\widetilde{p}(w) = v$.

The choice of the function w_0 is carried out in step i) of the following elaboration. Step ii) and iii) give the details of the extension of w_0 to a solution of $\widetilde{p}(w) = v$ on the positive real line, whereas step iv) extends w_0 for negative time.

i) Let $f \in \mathcal{C}^\infty[0,L]$ satisfy

$$\sum_{i=0}^{l} a_i f^{(i)}(t) = v(t), \;\; t \in [0,L], \;\; f^{(\nu)}(L) = 0 \text{ for } \nu = 0, \dots, l-1.$$

(In the case $l = 0$ one has no freedom for the initial conditions. In this case the rest of the proof in ii) and iii) works analogously.) Let $g \in \mathcal{C}^\infty[0,L]$ be such that $g|_{[0,L-1]} = 0$ and $g|_{[L-0.5,L]} = 1$. Put $w_0 := fg \in \mathcal{C}^\infty[0,L]$. Then $w_0|_{[0,L-1]} = 0$ and

$$w_0^{(\nu)}(L) = f^{(\nu)}(L) = \begin{cases} 0 & \text{for } \nu = 0, \dots, l-1 \\ v^{(\nu-l)}(L) - \sum_{i=0}^{l-1} a_i w_0^{(\nu-l+i)}(L) & \text{for } \nu \geq l. \end{cases}$$

ii) Let $\bar{w}_1 \in \mathcal{C}^\infty[L, L+1]$ fulfill

$$\sum_{i=0}^{l} a_i \bar{w}_1^{(i)}(t) = v(t) - \sum_{j=1}^{L} \widetilde{p_j}(w_0)(t-j), \;\; t \in [L, L+1], \;\; \bar{w}_1^{(\nu)}(L) = 0 \text{ for } \nu = 0, \dots, l-1.$$

By differentiation one checks that $\bar{w}_1^{(\nu)}(L) = w_0^{(\nu)}(L)$ for all $\nu \in \mathbb{N}_0$ and thus $w_1 := w_0 \wedge_L \bar{w}_1 \in \mathcal{C}^\infty[0, L+1]$ fulfills $\sum_{j=0}^{L} \widetilde{p_j}(w_1)(t-j) = v(t)$ for $t \in [L, L+1]$.

iii) Inductively, if $w_k \in \mathcal{C}^\infty[0, L+k]$ satisfies $\sum_{i=0}^{l} a_i w_k^{(i)}(t) = v(t) - \sum_{j=1}^{L} \widetilde{p_j}(w_k)(t-j)$ for $t \in [L, L+k]$, then take the solution $\bar{w}_{k+1} \in \mathcal{C}^\infty[L+k, L+k+1]$ of the ODE

$$\sum_{i=0}^{l} a_i \bar{w}_{k+1}^{(i)}(t) = v(t) - \sum_{j=1}^{L} \widetilde{p_j}(w_k)(t-j), \;\; t \in [L+k, L+k+1]$$

with initial conditions $\bar{w}_{k+1}^{(\nu)}(L+k) = w_k^{(\nu)}(L+k)$ for $\nu = 0, \dots, l-1$. From this we obtain again by differentiation $\bar{w}_{k+1}^{(\nu)}(L+k) = w_k^{(\nu)}(L+k)$ for all $\nu \in \mathbb{N}_0$ and therefore we get a solution $w_{k+1} := w_k \wedge_{L+k} \bar{w}_{k+1} \in \mathcal{C}^\infty[0, L+k+1]$. Hence we can construct a function $w_+ \in \mathcal{C}^\infty[0,\infty)$ which satisfies (A.1) for $t \geq L$.

iv) Let $\bar{w}_{-1} \in \mathcal{C}^\infty[-1,0]$ be satisfying

$$\sum_{i=0}^{r} b_i \bar{w}_{-1}^{(i)}(t) = v(t+L) - \sum_{j=0}^{L-1} \widetilde{p_j}(w_+)(t+L-j),\ \ \bar{w}_{-1}^{(\nu)}(0) = 0 \text{ for } \nu = 0, \dots, r-1$$

for $t \in [-1,0]$. Then $\bar{w}_{-1}^{(\nu)}(0) = 0$ for all $\nu \in \mathbb{N}_0$ and the function $w_{-1} := \bar{w}_{-1} \wedge_0 w_+ \in \mathcal{C}^\infty[-1,\infty)$ satisfies (A.1) for $t \geq L-1$. In an analogous way as in iii) we can proceed inductively and find finally a solution $w \in \mathcal{C}^\infty(\mathbb{R})$ for $\widetilde{p}(w) = v$.

b) Put $p = q\phi^{-1}$ with $q \in \mathcal{R}$ and $\phi \in \mathbb{R}[s]$ and let $w \in \mathcal{C}^\infty(\mathbb{R})$ be given as in Prop. 2.10 b). It is easy to see that there exists $v \in \mathcal{C}^\infty(\mathbb{R})$ with $\widetilde{\phi}(v) = w$ and $v|_{[k,k+L]} = 0$. But then $0 = \widetilde{p}(w) = \widetilde{q}(v)$ and the proof of a) shows by proceeding step by step on the intervals $[j, j+1]$, that $v = 0$ and thus $w = 0$. □

The following two lemmata are used in the proof of Thm. 3.2. The first one states the interpolation property for polynomials: given a finite set of points in the complex plane, there exists a polynomial $f \in \mathbb{C}[s]$, such that a specified number of derivatives $f^{(\nu)}$ take prescribed values at those points. If the required situation is symmetric with respect to complex conjugation, one can find a real interpolation polynomial.

Lemma A.1 *Let $\alpha_1, \dots, \alpha_r \in \mathbb{C}\backslash\mathbb{R}$, $\alpha_{r+1} \dots, \alpha_{r+t} \in \mathbb{R}$, $k_1, \dots, k_{r+t} \in \mathbb{N}_0$, $c_{j\nu} \in \mathbb{C}$ for $j = 1, \dots, r$ and $\nu = 0, \dots, k_j$ and $c_{j\nu} \in \mathbb{R}$ for $j = r+1, \dots, r+t$ and $\nu = 0, \dots, k_j$. Then there exists a unique $f \in \mathbb{R}[s]$ satisfying*

$$\begin{aligned} &\deg f \leq N := 2\textstyle\sum_{j=1}^{r}(k_j+1) + \sum_{j=r+1}^{r+t}(k_j+1) - 1 \\ &f^{(\nu)}(\alpha_j) = c_{j\nu} \text{ for } j = 1, \dots, r+t,\ \nu = 0, \dots, k_j \\ &f^{(\nu)}(\overline{\alpha_j}) = \overline{c_{j\nu}} \text{ for } j = 1, \dots, r,\ \nu = 0, \dots, k_j. \end{aligned}$$

Proof: The existence and uniqueness of $f \in \mathbb{C}[s]$ with the desired properties can be found e. g. in [5, p. 37]. But this already implies $f \in \mathbb{R}[s]$, since with $f = \sum_{j=0}^{N} f_j s^j \in \mathbb{C}[s]$ also $\bar{f} = \sum_{j=0}^{N} \bar{f}_j s^j$ fulfills the above requirements. □

The second lemma is just a rather specific calculation. It is used to show that the interpolation requirements given in (3.8) can be satisfied by one polynomial $f \in \mathbb{R}[s]$.

Lemma A.2 *Let $K \in \mathbb{N}_0$ and a_j, b_j, p_j, $q_j \in \mathbb{C}$ for $j = 0, \dots, K$. Let $p_0 \neq 0 \neq q_0$ and*

$$\sum_{m=0}^{n} \binom{n}{m} b_m q_{n-m} = -\sum_{m=0}^{n} \binom{n}{m} a_m p_{n-m} \text{ for } n = 0, \dots, K. \tag{A.2}$$

Put $f_n := q_0^{-1}[a_n - \sum_{m=0}^{n-1} \binom{n}{m} q_{n-m} f_m]$ for $n = 0, \dots, K$. Then also the recursion $f_n = -p_0^{-1}[b_n + \sum_{m=0}^{n-1} \binom{n}{m} p_{n-m} f_m]$ is valid for $n = 0, \dots, K$.

PROOF: For $n = 0$ it is $b_0 q_0 = -a_0 p_0$, hence $f_0 = \frac{a_0}{q_0} = -\frac{b_0}{p_0}$.

Suppose, the claim holds true for $f_0, \ldots, f_n$, $n < K$. Then one calculates

$$\begin{aligned}
q_0 f_{n+1} &= a_{n+1} - \sum_{m=0}^{n} \binom{n+1}{m} q_{n+1-m} f_m \\
&= a_{n+1} + \sum_{m=0}^{n} \binom{n+1}{m} q_{n+1-m} \left[\frac{1}{p_0} \left(b_m + \sum_{k=0}^{m-1} \binom{m}{k} p_{m-k} f_k \right) \right] \\
&= a_{n+1} + \frac{1}{p_0} \left[\sum_{m=0}^{n} \binom{n+1}{m} q_{n+1-m} b_m + \sum_{m=1}^{n} \sum_{k=0}^{m-1} \binom{n+1}{m} \binom{m}{k} q_{n+1-m} p_{m-k} f_k \right] \\
&= \frac{1}{p_0} \left[-b_{n+1} q_0 - \sum_{m=0}^{n} \binom{n+1}{m} a_m p_{n+1-m} + \sum_{k=0}^{n-1} \sum_{m=k+1}^{n} \binom{n+1}{m} \binom{m}{k} q_{n+1-m} p_{m-k} f_k \right] \\
&= -\frac{1}{p_0} \left[b_{n+1} q_0 + \sum_{m=0}^{n} \binom{n+1}{m} a_m p_{n+1-m} - \sum_{m=1}^{n} \sum_{k=0}^{m-1} \binom{n+1}{m} \binom{m}{k} p_{n+1-m} q_{m-k} f_k \right] \\
&= -\frac{1}{p_0} \left[b_{n+1} q_0 + q_0 \sum_{m=0}^{n} \binom{n+1}{m} p_{n+1-m} \frac{1}{q_0} \left(a_m - \sum_{k=0}^{m-1} \binom{m}{k} q_{m-k} f_k \right) \right] \\
&= -\frac{q_0}{p_0} \left[b_{n+1} + \sum_{m=0}^{n} \binom{n+1}{m} p_{n+1-m} f_m \right]
\end{aligned}$$

where the fourth equation follows from (A.2) and the fifth one from

$$\begin{aligned}
\sum_{m=k+1}^{n} \binom{n+1}{m} \binom{m}{k} q_{n+1-m} p_{m-k} &= \sum_{l=k+1}^{n} \binom{n+1}{n+1+k-l} \binom{n+1+k-l}{k} q_{l-k} p_{n+1-l} \\
&= \sum_{l=k+1}^{n} \binom{n+1}{l} \binom{l}{k} q_{l-k} p_{n+1-l}.
\end{aligned}$$

□

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