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Continuous State Representations for AR Systems*

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Abstract. In this paper we consider in a behavioral setting the subclass of discrete-time, linear, finite-dimensional systems, which can be represented by autoregressive (AR) equations. It will be shown that, with respect to the convergence of all coefficients in an AR representation, there exist continuously dependent input–state–output (i/s/o) representations, under the condition that some specified degree remains constant. This continuous i/s/o representation is given by the Fuhrmann realization.

Key words. Linear systems, Realization theory, Polynomial matrix representations, Continuity of systems.

1. Introduction

In many areas of control theory the question of continuity plays an important role, e.g. in the issue of structural stability or in robust control. For the formulation of continuity results there were introduced various kinds of topologies, metrics, or notions of convergence on the spaces of systems under consideration. One natural choice for the notion of continuity is the convergence of the corresponding system's parameters. But such a formulation is possible only if there are no structural changes within the class of systems under consideration, e.g. if the state-space dimension remains fixed. If one is interested in larger class of systems, it is more adequate to consider the systems as input–output operators between appropriate function spaces and to topologize accordingly in an operator theoretical way, as is done for, e.g. the gap topology.

In this paper we want to study continuity properties for a special class of systems within the first approach, namely continuity of realizations with respect to the parameters of the system. The systems under consideration will be linear, time-invariant, discrete-time autoregressive (AR) systems over the time-axis $\mathbb{Z}$ in the behavioral setting; see [W2]. We will investigate the question, under which conditions have a convergent sequence of AR-systems given by matrices $R^l \in \mathbb{R}[s]^{p\times q}$, $l \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, a converging sequence of input–state–output (i/s/o) representations (A^l, B^l, C^l, D^l). The underlying notion of continuity is the

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convergence of all coefficients in the matrices. The precise formulation of all these objects will be given below; see e.g. Definition 1.2.

For a special case this question is already answered. Precisely, if the matrices $R^l \in \mathbb{R}[s]^{p\times q}$ have full row rank for all $s \in \mathbb{C}$, they can be viewed (up to permutation of the columns) as coprime factorizations $R^l = [P^l, Q^l]$ of proper rational transfer matrices $(Q^l)^{-1}P^l$. From realization theory it is known that there exists a natural homeomorphism between the space of all proper transfer matrices of McMillan degree r and the quotient space of minimal state-space systems of dimension r under similarity; see [MH], [BD], and [H]. This result also includes the existence of minimal state-space realizations, which are continuous with respect to the Euclidian topology, hence with respect to parameter convergence. In this situation, the space of transfer matrices is topologized as a space of maps from the Riemann sphere to a Grassmannian. But it is possible to show the coincidence of this topology on the space of proper transfer matrices of degree r with the topology of convergence of all coefficients in a suitable coprime polynomial factorization [G]. In this sense, the present paper generalizes the above mentioned question of continuous realizations to the behavioral and nonminimal setting.

On the other hand, it can also be proven the coincidence of the topology on the space of transfer matrices with the gap or graph topology as defined in [ZE] and [VSF] as well as with the pointwise-gap metric introduced in [QD]; see [DGS] and [G]. This is due to the fact that the fixed McMillan degree prevents pole–zero cancellations. These last mentioned topologies were introduced in the theory of robust control as topologies for input–output operators. They are, of course, applicable to much larger spaces of transfer functions. But, as indicated above, in the very special case of fixed McMillan degree all these natural notions of convergence of systems coincide.

In the behavioral setting are—up to now—two results concerning continuity of systems. In order to represent these results we need some notions, which are also basic to the present paper. Thus we will firstly introduce these concepts in detail before coming back to the issue of continuity.

The systems under consideration are discrete-time, time-invariant and linear. They are given by triples $(\mathbb{Z}, \mathbb{R}^q, \mathscr{B})$, where $\mathbb{Z}$ is the time-axis, $\mathbb{R}^q$ the signal space and $\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{Z}} := \mathbb{L}^q$ the *behavior*, which is assumed to be linear and also shift-invariant, i.e. $\sigma(\mathscr{B}) = \mathscr{B}$ with the backward-shift $\sigma w(k) = w(k+1)$ for $w \in \mathbb{L}^q$. Hence $\mathscr{B}$ is just a set of trajectories. For a detailed introduction in these type of systems see e.g. [W2]. A subclass of such systems are the *autoregressive systems* (AR systems) $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathscr{B})$, where $\mathscr{B}$ is given as

$$\mathscr{B} = \ker R(\sigma) = \{w \in \mathbb{L}^q | R(\sigma)w = 0\} \tag{1.1}$$

with a suitable matrix $R \in \mathbb{R}[s, s^{-1}]^{p\times q}$. It is shown in [W2, Prop. 4.1 A] that a system has an AR representation as in (1.1) iff $\mathscr{B}$ is a closed subspace in $\mathbb{L}^q$, equipped with the topology of pointwise convergence. If such an AR-representation $\mathscr{B} = \ker R(\sigma)$ exists, one can assume rk $R = p$, so that, moreover, the representation becomes unique up to left unimodular factors in $Gl_p(\mathbb{R}[s, s^{-1}])$; see [W3, Prop. III.3]. Since $\sigma\colon \mathbb{L}^q \to \mathbb{L}^q$ is an automorphism, an AR-representation R

can always be chosen in $\mathbb{R}[s]^{p\times q}$, but for the uniqueness one needs in fact the unimodular factors over $\mathbb{R}[s, s^{-1}]$.

The representation of systems in the above sense goes one step further. It is shown in [W2, Thm. 4.3] that for an AR system $\ker R(\sigma)$ with $R \in \mathbb{R}[s, s^{-1}]^{p\times q}$ the external variable w (see (1.1)) can be partitioned into $m = q - p$ input variables u and p output variables y, i.e. $w = \tau(u^t, y^t)^t$ with a permutation τ, so that there exists $r \in \mathbb{N}$ and matrices $(A, B, C, D) \in \Sigma(r, m, p) := \mathbb{R}^{r^2+rm+pr+pm}$ such that

$$\ker R(\sigma) = \{\tau(u^t, y^t)^t \in \mathbb{L}^q \mid \exists x \in \mathbb{L}^r: \sigma x = Ax + Bu, y = Cx + Du\}. \tag{1.2}$$

Hence the given behavior $\ker R(\sigma)$ is—up to the permutation τ—just the so-called *external behavior* of an *input–state–output system* (i/s/o system). Note that the permutation τ just extracts a full size minor of R with maximal degree, i.e. $R\tau = [P, Q]$ with $Q^{-1}P$ proper. On the other hand, it can easily be seen that for $(A, B, C, D) \in \Sigma(r, m, p)$ the external behavior

$$\mathscr{B}(A, B, C, D) := \{(u^t, y^t)^t \in \mathbb{L}^{m+p} \mid \exists x \in \mathbb{L}^r: \sigma x = Ax + Bu, y = Cx + Du\} \tag{1.3}$$

is a closed subspace of $\mathbb{L}^{m+p}$ [W2, Prop. 4.1.C] and that every AR representation $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, D)$ fulfills $\det Q \neq 0$ and $Q^{-1}P \in \mathbb{R}(s)^{p\times m}$ being proper [W3, Prop. X.3], [W2, Prop. 4.6]. It should be mentioned that even in the case $R = [P, Q]$ with $Q^{-1}P$ being proper, $\deg \det Q$ is in general not the McMillan degree of the system associated with $\ker[P(\sigma), Q(\sigma)]$; see [W3, p. 276] and Remark 2.6 of this paper.

For i/s/o representations of behaviors a minimality criterion holds, similar to that of classical realization theory for transfer functions. In the special case considered the following is valid.

Remark 1.1. (a) (A, B, C, D) is a minimal i/s/o representation of $\ker R(\sigma)$ with respect to the state-space dimension iff (A, C) is observable in the usual sense and $[A, B]$ has full row rank [W2, Thm. 4.2].

(b) As in the case of transfer functions, a minimal i/s/o representation is unique up to similarity transformation [W3, Prop. IX.8].

In order to formulate the continuity results for autoregressive systems, we have to make precise the notion of convergence of all coefficients in a polynomial matrix.

Definition 1.2. Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. For $l \in \overline{\mathbb{N}}$ let $R^l = \sum_{i=0}^{N_l} R_i^l s^i \in \mathbb{R}[s]^{p\times q}$. Define $R^l \to R^\infty$ for $l \to \infty$ if there exists $N \in \mathbb{N}$ such that $N_l \leq N$ for all $l \in \overline{\mathbb{N}}$ and $R_i^l \to R_i^\infty$ for all $i = 0, \ldots, N$ in the Euclidian topology of $\mathbb{R}^{p\times q}$ (where we put $R_i^l = 0$ for $N_l < i \leq N$).

This notion of convergence was introduced in [NW]. Furthermore, using the usual concept of convergence of closed linear subsets $\mathscr{B}$ of the topological space $\mathbb{L}^q$, it was proven in [NW] that: (a) convergence of full rank polynomial matrices in the sense of Definition 1.2 implies convergence of the associated behaviors; (b) convergence of closed behaviors in $\mathbb{L}^q$ with bounded total lag implies the existence of converging AR-matrix representations [NW, p. 152]. Moreover, in [N, p. 26] it is shown that the convergence of matrices (A, B, C, D) of an i/s/o system as in (1.3) implies the convergence of the associated external behaviors if the limit system is observable.

In this paper we will consider the other direction, namely the problem of existence of continuous i/s/o representations for AR systems: given $R^l \in \mathbb{R}[s]^{p\times q}$ for $l \in \overline{\mathbb{N}}$ with $R^l \to R^\infty$ in the sense of Definition 1.2, does there exist a permutation τ of the external variables and matrices (A^l, B^l, C^l, D^l) in some $\Sigma(r, m, p)$ so that $\ker(R^l\tau)(\sigma) = \mathscr{B}(A^l, B^l, C^l, D^l)$ and $(A^l, B^l, C^l, D^l) \to (A^\infty, B^\infty, C^\infty, D^\infty)$? By the above, the existence of i/s/o representations implies at once that $R^l\tau = [P^l, Q^l]$ with $\det Q^l \not\equiv 0$ and $(Q^l)^{-1}P^l \in \mathbb{R}(s)^{p\times(q-p)}$ proper for $l \in \overline{\mathbb{N}}$. It will be shown that in this situation there exists a continuous i/s/o representation (A^l, B^l, C^l, D^l) iff $\deg\det Q^l = \deg\det Q^\infty$ for almost all $l \in \overline{\mathbb{N}}$. Moreover, if this is the case, the representation can always be chosen as observable. It is given by the Fuhrmann realization.

In the case of *continuous-time* autoregressive systems, another procedure for continuous realizations was given in [RR1] and [RR2]. Firstly, in [RR1] a smooth compactification of all autoregressive systems of degree r is presented. After this, in [RR2] a homeomorphism between this compact space and an orbit space of degree one realizations is constructed. In Remark 3.5(d) we will sketch, how the continuous i/s/o representation constructed in this paper also works in the continuous-time case and leads in the special case under consideration to an explicit version of the homeomorphism of [RR2].

2. I/s/o Representations for AR Systems

We start with the construction of an i/s/o representation for $\ker R(\sigma)$, where $R = [P, Q] \in \mathbb{R}[s]^{p\times(m+p)}$. This is only a slight modification of the representation given in [W1, p. 577]. The construction is mainly a reorganization of all the finitely many data involved in the polynomial matrix R in order to get a representation of $\ker R(\sigma)$ as a singular system, which, in some cases, gives an i/s/o representation. Let $Q \in \mathbb{R}[s]^{p\times p}$, $P \in \mathbb{R}[s]^{p\times m}$ be given by

$$Q(s) = \operatorname{diag}(s^{n_1}, \dots, s^{n_p})\hat{Q} + (q_{ij})_{i,j=1,\dots,p} \quad \text{with} \quad q_{ij} = \sum_{k=0}^{n_i-1} q_{ij}^k s^k \quad \text{and} \quad \hat{Q} \in \mathbb{R}^{p\times q}, \tag{2.1}$$

$$P(s) = \operatorname{diag}(s^{n_1}, \dots, s^{n_p})\hat{P} + (p_{ij})_{\substack{i=1,\dots,p\\ j=1,\dots,m}} \quad \text{with} \quad p_{ij} = \sum_{k=0}^{n_i-1} p_{ij}^k s^k \quad \text{and} \quad \hat{P} \in \mathbb{R}^{p\times m}. \tag{2.2}$$

Denote by $\mathrm{h}[P, Q]$ the highest coefficient matrix of $[P, Q]$. Then $[\hat{P}, \hat{Q}] = \mathrm{h}[P, Q]$ iff $n_1, \dots, n_p$ are the row degrees of $[P, Q]$. For the following construction it is also allowed that n_i is larger than the ith row degree of $[P, Q]$. Hence there might be zero rows in the matrix $[\hat{P}, \hat{Q}]$, or, in an extreme case, $[\hat{P}, \hat{Q}] = 0$. Suppose $n_i > 0$ for all i. Furthermore, suppose that there exists $D \in \mathbb{R}^{p\times m}$ such that

$\hat{P} = \hat{Q}D$. This is for instance the case, if $[P, Q]$ is row-proper (i.e. $\operatorname{rk} \mathrm{h}[P, Q] = p$) with row degrees $n_1, \ldots, n_p$, and $Q^{-1}P$ is proper, but also, if $[\hat{P}, \hat{Q}] = 0$.

Put $r := \sum_{i=1}^{p} n_i$, $\hat{Q} = (\hat{q}_{ij})$, $\hat{P} = (\hat{p}_{ij})$ and define

$$E = (E_{ij})_{i,j=1,\ldots,p} \in \mathbb{R}^{r\times r}, \quad \text{with} \quad E_{ij} = \begin{cases} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & 1 & \\ 0 & & -\hat{q}_{ii} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} & \text{for } i = j, \\ \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\hat{q}_{ij} \end{bmatrix} \in \mathbb{R}^{n_i \times n_j} & \text{for } i \neq j, \end{cases} \tag{2.3}$$

$$A = (A_{ij})_{i,j=1,\ldots,p} \in \mathbb{R}^{r\times r}, \quad \text{with} \quad A_{ij} = \begin{cases} \begin{bmatrix} 0 & & & q_{ii}^{0} \\ 1 & & & q_{ii}^{1} \\ & \ddots & & \vdots \\ & & 1 & q_{ii}^{n_i-1} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} & \text{for } i = j, \\ \begin{bmatrix} 0 & \cdots & 0 & q_{ij}^{0} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & q_{ij}^{n_i-1} \end{bmatrix} \in \mathbb{R}^{n_i \times n_j} & \text{for } i \neq j. \end{cases} \tag{2.4}$$

Further let

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_p \end{bmatrix} \in \mathbb{R}^{r\times m} \quad \text{with} \quad B_i = (b_{ij}^{k})_{\substack{k=0,\ldots,n_i-1 \\ j=1,\ldots,m}}, \quad \text{where} \quad b_{ij}^{k} = p_{ij}^{k} - \sum_{l=1}^{p} q_{il}^{k} d_{lj}, \tag{2.5}$$

$$C = [C_1, \ldots, C_p] \in \mathbb{R}^{p\times r} \quad \text{with} \quad C_i = [0, \ldots, 0, e_i] \in \mathbb{R}^{p \times n_i}, \tag{2.6}$$

where e_i denotes the ith standard basis vector in $\mathbb{R}^p$.

Proposition 2.1. *With the above notations it holds that*

$$\ker[P(\sigma), Q(\sigma)] = \{(u^t, y^t)^t \in \mathbb{L}^{m+p} \mid \exists x \in \mathbb{L}^r \colon E\sigma x = Ax + Bu,\ y = Cx - Du\}.$$

In particular, if $\hat{Q} \in Gl_p(\mathbb{R})$, *then* $E \in Gl_r(\mathbb{R})$ *and* $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(E^{-1}A, E^{-1}B, C, -D)$. *In this case, the i/s/o representation is always observable and it is minimal iff additionality* $\operatorname{rk}[P(0), Q(0)] = p$ *holds.*

Note that in the case $Q^{-1}P$ proper and $\operatorname{rk} \mathrm{h}[P, Q] = p$, this representation is similar to that given in [W1, p. 577], which has the form $(AE^{-1}, B, \hat{Q}^{-1}C, -D)$.

Proof. ($\subseteq$) Let $(u^t, y^t)^t \in \mathbb{L}^{m+p}$ with $P(\sigma)u + Q(\sigma)y = 0$. Put $x = (x_{11}, \ldots, x_{1n_1}, \ldots, x_{p1}, \ldots, x_{pn_p})^t$ with $x_{in_i} = y_i + \sum_{k=1}^{m} d_{ik}u_k$ for $i = 1, \ldots, p$ and $\sigma^k x_{ik} =$

$\sum_{l=0}^{k-1}\sum_{j=1}^{p} q_{ij}^{l}\sigma^{l}x_{jn_j} + \sum_{l=0}^{k-1}\sum_{j=1}^{m} b_{ij}^{l}\sigma^{l}u_j$ for $i = 1, \ldots, p$ and $k = 1, \ldots, n_i - 1$. Then it holds by construction $y = Cx - Du$ and one also checks $E\sigma x = Ax + Bu$.

($\supseteq$) From the system equation it follows by induction $\sigma^k x_{ik} = \sum_{l=0}^{k-1}\sum_{j=1}^{p} q_{ij}^{l}\sigma^{l}x_{jn_j} + \sum_{l=0}^{k-1}\sum_{j=1}^{m} b_{ij}^{l}\sigma^{l}u_j$ for $i = 1, \ldots, p$ and $k = 1, \ldots, n_i - 1$. Moreover $-\sum_{j=1}^{p}\hat{q}_{ij}\sigma x_{jn_j} = x_{i,n_i-1} + \sum_{j=1}^{p} q_{ij}^{n_i-1}x_{jn_j} + \sum_{j=1}^{m} b_{ij}^{n_i-1}u_j$. This and the output equations $y_j = x_{jn_j} - \sum_{k=1}^{m} d_{jk}u_k$ for $j = 1, \ldots, p$ yield by straightforward calculation $P(\sigma)u + Q(\sigma)y = 0$.

If $\hat{Q} \in Gl_p(\mathbb{R})$, then $\mathrm{rk}[(sE - A)^t, C^t]^t = r$ for all $s \in \mathbb{C}$, which can be seen easily from the matrices. Moreover, it is $\mathrm{rk}[E^{-1}A, E^{-1}B] = r$ iff $\mathrm{rk}[Q(0), P(0) - Q(0)D] = p$ and this is equivalent to $\mathrm{rk}[P(0), Q(0)] = p$. With Remark 1.1 this yields the desired result. ■

Remark 2.2. (a) Note that this representation holds also in the case where the n_i are not the row degrees of $[P, Q]$. We need only the assumption $\mathrm{Im}\,\hat{P} \subseteq \mathrm{Im}\,\hat{Q}$, which is, e.g. trivially fulfilled, if both matrices are zero. In this case the above given representation is highly non-minimal and, of course, not an i/s/o representation.

(b) It is worth noting that the construction in Proposition 2.1 leads also to an i/s/o representation in the case that $\mathrm{h}[P, Q] = [0, I]$ and some (but not all!) of the n_i's are zero. In this case one has to omit just those blocks, whose size involves n_i. This holds true, since in the case $\mathrm{h}[P, Q] = [0, I]$ the equality $n_i = 0$ leads to an equation $y_i = 0$ in $P(\sigma)u + Q(\sigma)y = 0$ as well as in $y = Cx$.

As a consequence from this special representation one gets the following.

Corollary 2.3. *Let $R = [P, Q] \in \mathbb{R}[s]^{p\times(m+p)}$ with $Q^{-1}P \in \mathbb{R}(s)^{p\times m}$ proper and let $(A, B, C, D) \in \Sigma(r, m, p)$ be an arbitrary i/s/o representation of* $\ker R(\sigma)$. *Then $\chi_A(s) := \det(sI - A) = v(s)\det Q(s)$ for some $v \in \mathbb{R}[s, s^{-1}]$. If* $\mathrm{rk}\, R(0) = p$, *then $v \in \mathbb{R}[s]$. If (A, B, C, D) is a minimal representation of* $\ker R(\sigma)$, *then $v(s) = \alpha s^k$ for some $\alpha \in \mathbb{R}\backslash\{0\}$ and $k \in \mathbb{Z}$. If additionally* $\mathrm{rk}\, R(0) = p$, *then $k = 0$.*

Proof. We start with an arbitrary i/s/o representation (A, B, C, D) of $\ker R(\sigma)$.

1. If (A, C) is not observable, then there exists $T \in Gl_r(\mathbb{R})$ such that

$$(TAT^{-1}, TB, CT^{-1}, D) = \left(\begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1, 0], D\right)$$

with $(A_1, C_1) \in \mathbb{R}^{t^2+pt}$ for some $t \in \mathbb{N}$ being observable. One checks $\mathscr{B}(A, B, C, D) = \mathscr{B}(A_1, B_1, C_1, D)$ and we have $\chi_A = \chi_{A_1}\chi_{A_4}$.

2. Let $A_1 \in \mathbb{R}^{t\times t}$. If $\mathrm{rk}[A_1, B_1] < t$, then in particular the system (A_1, B_1, C_1, D) is not reachable and hence there exists $S \in Gl_t(\mathbb{R})$ such that

$$(SA_1S^{-1}, SB_1) = \left(\begin{bmatrix} A_1' & A_2' \\ 0 & A_4' \end{bmatrix}, \begin{bmatrix} B_1' \\ 0 \end{bmatrix}\right)$$

with $\det A_4' = 0$ and (A_1', B_1') reachable. We can assume

$$\hat{A}_4 = \begin{bmatrix} A'' & 0 \\ 0 & J \end{bmatrix}$$

with J in nilpotent Jordan-form. Since it follows from the system equation $\sigma x = Jx$, that $x = 0$, we can omit the nilpotent part of the system and thus we get $\mathscr{B}(A_1, B_1, C_1, D) = \mathscr{B}(\tilde{A}, \tilde{B}, \tilde{C}, D)$ with appropriate matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, which is still observable and where $[\tilde{A}, \tilde{B}]$ has full row rank. The construction yields $\chi_{A_1} = \chi_J \chi_{\tilde{A}}$.

Both parts together lead to $\chi_A = v\chi_{\tilde{A}}$ with some $v \in \mathbb{R}[s]$ where $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ is a minimal i/s/o representation of $\ker R(\sigma)$.

3. Since a minimal representation is unique up to similarity (Remark 1.1), we can take a representation as constructed in Proposition 2.1 if it is minimal. The state equations resulting from the matrices (E, A, B, C, D) given in (2.3)–(2.5) can be permuted to

$$\begin{bmatrix} I & 0 \\ 0 & -\hat{Q} \end{bmatrix} \sigma x = \begin{bmatrix} A_1 & A_2 \\ A_3 & \tilde{Q} \end{bmatrix} x + \tilde{B}u,$$

with

$$A_1 = \operatorname{diag}_{i=1,\dots,p} \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{(n_i-1)\times(n_i-1)} \in \mathbb{R}^{(r-p)\times(r-p)},$$

$$A_2 = \begin{bmatrix} A_{21} \\ \vdots \\ A_{2p} \end{bmatrix} \in \mathbb{R}^{(r-p)\times p},$$

where

$$A_{2i} = (q_{ij}^k)_{\substack{k=0,\dots,n_i-2 \\ j=1,\dots,p}} \in \mathbb{R}^{(n_i-1)\times p}, \qquad A_3 = [e_{n_1-1}, e_{n_1+n_2-2}, \dots, e_{r-p}]^t \in \mathbb{R}^{p\times(r-p)},$$

and $\tilde{Q} = (q_{ij}^{n_i-1}) \in \mathbb{R}^{p\times p}$ (here e_j denotes the jth standard-basis vector of $\mathbb{R}^{r-p}$). Then

$$\det \begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & -s\hat{Q} - \tilde{Q} \end{bmatrix} = \det(sI - A_1) \det(-s\hat{Q} - \tilde{Q} - A_3(sI - A_1)^{-1}A_2).$$

From

$$(-A_3(sI - A_1)^{-1}A_2)_{ij} = -[s^{-(n_i-1)}, \dots, s^{-1}][q_{ij}^0, \dots, q_{ij}^{n_i-2}]^t = -\sum_{k=0}^{n_i-2} q_{ij}^k s^{-(n_i-1)+k}$$

we get

$$\begin{aligned} &\det(sI - A_1) \det(-s\hat{Q} - \tilde{Q} - A_3(sI - A_1)^{-1}A_2) \\ &\quad = s^{r-p} \det\left(-s\hat{Q} - \tilde{Q} - \left(s^{-(n_i-1)} \sum_{k=0}^{n_i-2} q_{ij}^k s^k \right)_{i,j=1,\dots,p} \right) = \det(-Q). \end{aligned}$$

Therefore the representation constructed in the proposition fulfills $\det(sE - A) = \pm \det Q(s)$. This representation is minimal iff $\operatorname{rk} R(0) = p = \operatorname{rk} \hat{Q}$. Now we get the corollary:

(a) If R is arbitrary, then there exists $U \in Gl_p(\mathbb{R}[s, s^{-1}])$ such that rk $UR(0) = p = \text{rk } \mathsf{h}[UR]$. It holds that $\det U(s) = \alpha s^k$ for some $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}\backslash\{0\}$. Thus with the minimal representation $(E^{-1}A, E^{-1}B, C, -D)$ for UR as constructed in Proposition 2.1 it follows that $\det Q(s) = \pm\alpha^{-1}s^{-k} \det E \det(sI - E^{-1}A)$. With parts 1 and 2 this yields the first part of the corollary.

(b) If rk $R(0) = p$, then the transformation U in (a) fulfills $U \in Gl_p(\mathbb{R}[s])$, hence $\det U = \alpha$. This gives the other parts of the claim. ■

In the following we show that the classical *Fuhrmann realization* [F, Ch. I.10] applied to $R = [P, Q] \in \mathbb{R}[s]^{p\times(m+p)}$ with $Q^{-1}P$ proper yields an i/s/o representation of $\ker R(\sigma)$.

Firstly, if $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, D)$ for some (A, B, C, D), then $D = -Q^{-1}P(\infty)$. This can be seen from the construction in the proof of Corollary 2.3, the uniqueness of minimal representations, and the special construction in Proposition 2.1 for the case rk $\mathsf{h}[P, Q] = \text{rk}[\hat{P}, \hat{Q}] = p$ with $\hat{P}$, $\hat{Q}$ as (2.1), (2.2). One can also show that in general $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, D)$ implies $-Q^{-1}P(s) = C(sI - A)^{-1}B + D$.

Therefore, with $D = -Q^{-1}P(\infty)$ and $P' := P + QD$ it holds that $Q^{-1}P'$ is strictly proper and for $(A, B, C, D) \in \Sigma(r, m, p)$ we get

$$\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, D) \Leftrightarrow \ker[P'(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, 0). \quad (2.7)$$

Hence we can restrict ourselves to the case $R = [P, Q] \in \mathbb{R}[s]^{p\times(m+p)}$ with $Q^{-1}P$ strictly proper.

For the Fuhrmann realization we need the ring of Laurent series $\mathbb{R}^p((s^{-1})) = \{\sum_{i=L}^{\infty} \alpha_i s^{-i} | \alpha_i \in \mathbb{R}^p, L \in \mathbb{Z}\}$. Put

$$\Pi_-: \mathbb{R}^p((s^{-1})) \to \mathbb{R}^p((s^{-1})),$$

$$\sum_{i=L}^{\infty} \alpha_i s^{-i} \mapsto \sum_{i=1}^{\infty} \alpha_i s^{-i}$$

the projection onto the strict proper part of the series, and

$$\Pi_Q: \mathbb{R}^p[s] \to \mathbb{R}^p[s],$$

$$f \mapsto Q\Pi_- Q^{-1}f.$$

Since $\Pi_- Q^{-1}f = Q^{-1}f$ minus a polynomial, it is in fact $Q\Pi_- Q^{-1}f$ polynomial. Define $K_Q := \text{Im } \Pi_Q \subseteq \mathbb{R}^p[s]$. It is easy to see that $K_Q = \{f \in \mathbb{R}^p[s] | Q^{-1}f$ strictly proper$\}$. Moreover, K_Q is a vector space with $\dim K_Q = \deg \det Q$. This can be shown by transforming Q into Smith form $Q' = \text{diag}(q_1, \ldots, q_p)$. Then, by *Fuhrmann* [F, Thm. 4.11] it holds $\dim K_Q = \dim K_{Q'}$. The last expression is equal to $\deg \det Q'$, since for $f = (f_1, \ldots, f_p)^t \in \mathbb{R}^p[s]$ with $f_i = a_i q_i + b_i$, $a_i, b_i \in \mathbb{R}[s]$ and $\deg b_i < \deg q_i$ it follows that $\Pi_{Q'}f = \Pi_{Q'}(b_1, \ldots, b_p)^t = (b_1, \ldots, b_p)^t$.

Lemma 2.4. *Let* $Q \in \mathbb{R}[s]^{p\times p}$ *with row degrees* $n_1, \ldots, n_p \geq 0$ *and* $\mathsf{h}[Q] = I_p$. *Then* $K_Q = \{(f_1, \ldots, f_p)^t \in \mathbb{R}^p[s] | \deg f_i < n_i\}$.

Proof. Remember $Q^{-1} = (\det Q)^{-1} \operatorname{adj}(Q)$. Since $\mathsf{h}[Q] = I_p$, it holds

$$\deg \operatorname{adj}(Q)_{ii} = \sum_{l \neq i} n_l \qquad \text{and} \qquad \deg \operatorname{adj}(Q)_{ji} < \sum_{l \neq i} \eta_l \quad \text{for} \quad j \neq i. \tag{2.8}$$

For $f = \tilde{f} e_k \in \mathbb{R}^p[s]$, $\tilde{f} \in \mathbb{R}[s]$, it is $Q^{-1} f = \tilde{f} (\det Q)^{-1} (\operatorname{adj}(Q)_{1k}, \ldots, \operatorname{adj}(Q)_{pk})^t$ and thus

$$Q^{-1} f \text{ is strictly proper} \Leftrightarrow \deg \tilde{f} + \deg \operatorname{adj}(Q)_{kk} < \deg \det Q = \sum_{i=1}^{p} n_i \Leftrightarrow \deg \tilde{f} < n_k.$$

With $\dim K_Q = \sum_{i=1}^{p} n_i$ this yields the result. ■

Now we can formulate the Fuhrmann realization.

Proposition 2.5. *For* $[P, Q] \in \mathbb{R}[s]^{p \times (m+p)}$ *with* $\operatorname{rk} Q = p$ *and* $Q^{-1} P$ *strictly proper, put*

$$\begin{array}{lll} A\colon K_Q \to K_Q, & B\colon \mathbb{R}^m \to K_Q, & C\colon K_Q \to \mathbb{R}^p, \\ f \mapsto \Pi_Q(sf), & \xi \mapsto -P\xi, & f \mapsto (Q^{-1} f)_1, \end{array}$$

where $Q^{-1} f = \sum_{n=1}^{\infty} (Q^{-1} f)_n s^{-n}$. *Then it holds that* $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, 0)$.

Proof. 1. We consider first the case $\mathsf{h}[P, Q] = [0, I_p]$. Let $r = \sum_{i=1}^{p} n_i$ where $n_1, \ldots, n_p \geq 0$ are the row degrees of $[P, Q]$. We will show that the above defined operators have, with respect to a suitably chosen basis of K_Q, matrix representations, which are identical to the matrices constructed in Proposition 2.1. This works also in the case that some of the n_i are zero (see Remark 2.2(b)). In the case $r = 0$, the operators A, B, C vanish, whereas $D = 0$, so in this case there is nothing to prove. Let $r > 0$. By Lemma 2.4 we know that K_Q has the basis

$$\mathscr{A} = \{ f_{ij} := s^j e_i \mid i = 1, \ldots, p,\ j = 0, \ldots, n_i - 1 \},$$

where e_i denotes the ith standard-basis vector in $\mathbb{R}^p$. Then the following hold.

(a) For $j < n_i - 1$, $A(f_{ij}) = f_{i,j+1}$.
(b) Let $f = f_{i,n_i - 1} = s^{n_i - 1} e_i$. Then

$$\Pi_Q(sf) = Q \Pi_- Q^{-1}(s^{n_i} e_i) = Q \Pi_- (\det Q)^{-1} s^{n_i} (\operatorname{adj}(Q)_{1i}, \ldots, \operatorname{adj}(Q)_{pi})^t.$$

Since $\mathsf{h}[Q] = I_p$, (2.8) yields $\Pi_Q(sf) = s^{n_i} e_i - Q e_i$. With $Q = (q_{ki}) = (\delta_{ki} s^{n_k} + \sum_{l=0}^{n_k - 1} q_{ki}^l s^l)$ we get

$$\Pi_Q(sf) = -\sum_{k=1}^{p} \sum_{l=0}^{n_k - 1} q_{ki}^l s^l e_k = -\sum_{k=1}^{p} \sum_{l=0}^{n_k - 1} q_{ki}^l f_{kl}.$$

Thus, with respect to the basis $\mathscr{A}$ of K_Q, the image $\Pi_Q(sf)$ has—up to a minus sign—just the $(n_1 + \cdots + n_i)$th column of the matrix A given in (2.4) as coordinate vector.

The matrix representation of B with respect to the basis $\mathscr{A}$ of K_Q (and the standard basis of $\mathbb{R}^m$) is given—up to a minus sign—by the matrix B as in (2.5), since with $P = (\sum_{l=0}^{n_k - 1} p_{kj}^l s^l)_{kj}$ it holds that $B e_j = -P e_j = -\sum_{k=1}^{p} \sum_{l=0}^{n_k - 1} p_{kj}^l f_{kl}$.

Finally the matrix representation of the operator C is identical to the matrix C as in (2.6), since by (2.8) it follows $C(f_{ij}) = (Q^{-1}s^j e_i)_1 = e_i$ if $j = n_i - 1$ and 0 otherwise.

Taking into account the minus sign in the matrix E in (2.3), we get from Proposition 2.1 that $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, C, D)$.

2. Let $\mathsf{h}[P, Q] = [0, \hat{Q}]$ with $\hat{Q} \in Gl_p(\mathbb{R})$. Then $[P, Q\hat{Q}^{-1}]$ fulfills the requirements of Part 1 and for $f \in \mathbb{R}^p[s]$ it holds $\Pi_{Q\hat{Q}^{-1}}(f) = \Pi_Q(f)$. Therefore $K_{Q\hat{Q}^{-1}} = K_Q$. Furthermore $(\hat{Q}Q^{-1}f)_1 = \hat{Q}(Q^{-1}f)_1$. Thus, if $(A, B, C, 0)$ is the Fuhrmann realization of $\ker[P, Q\hat{Q}^{-1}]$, then $(A, B, \hat{Q}^{-1}C, 0)$ is that of $[P, Q]$. Since from $\ker[P(\sigma), Q\hat{Q}^{-1}(\sigma)] = \mathscr{B}(A, B, C, 0)$ it follows that $\ker[P(\sigma), Q(\sigma)] = \mathscr{B}(A, B, \hat{Q}^{-1}C, 0)$, and we get in fact an i/s/o representation of $\ker[P(\sigma), Q(\sigma)]$.

3. Now let $[P, Q]$ be arbitrary of full row rank and $U \in Gl_p(\mathbb{R}[s])$ such that $U[P, Q]$ is row-proper. Then $f \mapsto Uf$ defines an $\mathbb{R}$-isomorphism from K_Q to K_{UQ}. Furthermore, for $f \in K_Q$ it holds that $\Pi_{UQ}(sUf) = U\Pi_Q(sf)$ and $((UQ)^{-1}(Uf))_1 = (Q^{-1}f)_1$. This shows that if A, B, C are the operators associated with UR as in the proposition and A', B', C' are those associated with R, then $A = UA'U^{-1}$, $B = UB'$, $C = C'U^{-1}$. Since by part 2, (A, B, C) leads to an i/s/o representation of $\ker UR(\sigma) = \ker R(\sigma)$, the same holds for (A', B', C'). ■

Remark 2.6. One can check that the realization given in Proposition 2.1 as well as the coincidence with the Fuhrmann realization holds true also if we restrict to the time-axis $\mathbb{Z}_+$ instead of $\mathbb{Z}$ or if we replace the shift by differentiation and consider continuous-time systems with behaviors consisting of smooth functions only. In these cases, the Fuhrmann realization was already recovered in [KS, pp. 1175] as a realization, which works also in the behavioral context. In contrast to $\mathbb{Z}$, where σ is an automorphism, it holds for $\mathbb{Z}_+$ and for the continuous-time case, that $\dim \ker K_Q = \deg \det Q$ is just the McMillan degree of the system associated with $[P, Q] \in \mathbb{R}[s]^{p \times (m+p)}$ if $Q^{-1}P$ is proper. Hence the Fuhrmann realization is always minimal, as also mentioned in [KS, p. 1177]. In the case of full time-axis $\mathbb{Z}$, the McMillan degree, i.e. the minimal possible dimension of an i/s/o representation of a system given by $R \in \mathbb{R}[s, s^{-1}]^{p \times q}$ can be computed as follows: let

$$a = \sum_{j=l}^{L} a_j s^j \in \mathbb{R}[s, s^{-1}]^{1 \times \binom{q}{p}}$$

(with $a_l \neq 0 \neq a_L$) be the vector of all $p \times p$-minors of R. Then $\delta(R) := L - l$ is the McMillan degree of the system; see [W3, p. 276]. In the case $R \in \mathbb{R}[s]^{p \times q}$, one can see at once that the McMillan degree is given by the maximal degree of all $p \times p$-minors of R iff $\operatorname{rk} R(0) = p$.

3. Continuity of i/s/o Representations

We will show the continuity of the Fuhrmann realization in the case that $\deg \det Q^l = \deg \det Q^\infty$ for almost all $l \in \overline{\mathbb{N}}$. First we prove that this condition is in fact necessary for the existence of continuous i/s/o representations.

Proposition 3.1. *For $l \in \overline{\mathbb{N}}$ let $R^l = [P^l, Q^l] \in \mathbb{R}[s]^{p \times (m+p)}$ and suppose $R^l \to R^\infty$ in the sense of Definition 1.2. Further let $(A^l, B^l, C^l, D^l) \in \Sigma(r, m, p)$ for some $r \in \mathbb{N}$ with $\ker R^l(\sigma) = \mathscr{B}(A^l, B^l, C^l, D^l)$ and $(A^l, B^l, C^l, D^l) \to (A^\infty, B^\infty, C^\infty, D^\infty)$ in the Euclidian topology. Then it follows that $\deg\det Q^l = \deg\det Q^\infty$ for almost all $l \in \mathbb{N}$.*

Proof. In the above situation it is $(Q^l)^{-1}P^l$ proper [W2, Prop. 4.6]. In particular, $\det Q^l \neq 0$ for all $l \in \overline{\mathbb{N}}$. By Corollary 2.3 it holds that $\chi_{A^l} = v^l \det Q^l$ for some $v^l \in \mathbb{R}[s, s^{-1}]$ and for all $l \in \overline{\mathbb{N}}$. By the definition of $R^l \to R^\infty$ it holds that $\deg\det Q^l \leq N$ for some $N \in \mathbb{N}$ and all $l \in \mathbb{N}$ and $\det Q^l \to \det Q^\infty$. Write $v^l(s) = \sum_{i \geq k_l} v_i^l s^i$ with $v_{k_l}^l \neq 0$ and $\det Q^l(s) = \sum_{j \geq t_l} q_j^l s^j$ with $q_{t_l}^l \neq 0$. Then from $\chi_{A^l} = v^l \det Q^l \in \mathbb{R}[s]$ it follows that $k_l \geq -t_l \geq -N$. Since multiplication by s^N does not change continuity properties of polynomials, we can assume $v^l \in \mathbb{R}[s]$. It holds that $\chi_{A^l} \to \chi_{A^\infty}$, $\deg \chi_{A^l} = \deg \chi_{A^\infty}$ and $\det Q^l \to \det Q^\infty$, $\deg\det Q^\infty \leq \deg\det Q^l$ for almost all $l \in \mathbb{N}$. Let $\chi_{A^l}(s) = \prod_{i=1}^r (s - \alpha_i^l)$ for $l \in \overline{\mathbb{N}}$. Then $\alpha_i^l - \alpha_i^\infty$ for all $i = 1, \ldots, r$ (see e.g. [B, p. 22]). Suppose $\deg\det Q^\infty = n < \deg\det Q^l = N$ for infinitely many $l \in \mathbb{N}$. Since $\det Q^l$ divides χ_{A^l}, we can assume $\det Q^l = q^l \prod_{i=1}^N (s - \alpha_i^l)$ for $l \in \mathbb{N}$ and $\det Q^\infty = q^\infty \prod_{i=1}^n (s - \alpha_i^\infty)$. But then it follows again by [B, p. 22] $\alpha_i^l \to \alpha_i^\infty$ for $i = 1, \ldots, n$ and $\alpha_i^l \to \infty$ for $i = n + 1, \ldots, N$, which is a contradiction. ■

Before starting to show the continuity of the Fuhrmann realization in the above-mentioned case, we present the following lemma, which can easily be proven.

Lemma 3.2. *Let $p^l, q^l \in \mathbb{R}[s]$ for $l \in \overline{\mathbb{N}}$ with $p^l \to p^\infty$, $q^l \to q^\infty$ and $\deg q^l = r$ for all $l \in \overline{\mathbb{N}}$. Suppose $p^l = a^l q^l + b^l$ with $a^l, b^l \in \mathbb{R}[s]$ and $\deg b^l < r$. Then $a^l \to a^\infty$ and $b^l \to b^\infty$.*

Let us now assume $R^l \in \mathbb{R}[s]^{p \times (m+p)}$ for $l \in \overline{\mathbb{N}}$ with $R^l \to R^\infty$. After reordering of the columns we can assume $R^\infty = [P^\infty, Q^\infty]$ with $(Q^\infty)^{-1}P^\infty$ proper. Put $R^l = [P^l, Q^l]$ for $l \in \mathbb{N}$. In order to get continuous i/s/o representations for R^l, we need $(Q^l)^{-1}P^l$ to be proper and, by Proposition 3.1, $\deg\det Q^l = \deg\det Q^\infty$.

Proposition 3.3. *Let $[P^l, Q^l] \in \mathbb{R}[s]^{p \times (m+p)}$ for $l \in \overline{\mathbb{N}}$ with $[P^l, Q^l] \to [P^\infty, Q^\infty]$, $(Q^l)^{-1}P^l$ proper and $\deg\det Q^l = r$ and $l \in \overline{\mathbb{N}}$. Then*:

(a) *For $D^l = (Q^l)^{-1}P^l(\infty) \in \mathbb{R}^{p \times m}$, $l \in \overline{\mathbb{N}}$, it holds that $D^l \to D^\infty$.*
(b) *If $f^l \in \mathbb{R}[s]^p$ for $l \in \overline{\mathbb{N}}$ with $f^l \to f^\infty$, then $\Pi_{Q^l}(f^l) \to \Pi_{Q^\infty}(f^\infty)$. If $(Q^l)^{-1}f^l$ is strictly proper for all $l \in \overline{\mathbb{N}}$, it also holds $((Q^l)^{-1}f^l)_1 \to ((Q^\infty)^{-1}f^\infty)_1$.*

Proof. (a) We have $(Q^l)^{-1}P^l = (\det Q^l)^{-1} \operatorname{adj}(Q^l)P^l$. Let $(\operatorname{adj}(Q^l)P^l)_{ij} = a_{ij}^l \det Q^l + b_{ij}^l$ with $a_{ij}^l, b_{ij}^l \in \mathbb{R}[s]$ and $\deg b_{ij}^l < \deg\det Q^l$ for all $l \in \overline{\mathbb{N}}$ and $i = 1, \ldots, p$, $j = 1, \ldots, m$. Then, by Lemma 3.2, it follows $a_{ij}^l \to a_{ij}^\infty$, $b_{ij}^l \to b_{ij}^\infty$. The properness of $(Q^l)^{-1}P^l$ implies $a_{ij}^l \in \mathbb{R}$ and $(a_{ij}^l) = D^l$ for all $l \in \overline{\mathbb{N}}$. This proves (a).

(b) Let $\operatorname{adj}(Q^l)f^l = (g_1^l, \ldots, g_p^l)^t \in \mathbb{R}[s]^p$ and $g_i^l = a_i^l \det Q^l + b_i^l$ for all $l \in \overline{\mathbb{N}}$ and $i = 1, \ldots, p$ with $\deg b_i^l < \deg \det Q^l$. Then, again by Lemma 3.2, $a_i^l \to a_i^\infty$ and $b_i^l \to b_i^\infty$. We have

$$\begin{aligned}\Pi_{Q^l}(f^l) &= Q^l \Pi_-((\det Q^l)^{-1} \operatorname{adj}(Q^l) f^l) \\ &= f^l - Q^l(a_1^l, \ldots, a_p^l)^t \to f^\infty - Q^\infty(a_1^\infty, \ldots, a_p^\infty)^t = \Pi_{Q^\infty}(f^\infty).\end{aligned}$$

Suppose $(Q^l)^{-1}f^l$ being strictly proper for all $l \in \overline{\mathbb{N}}$. With the above notation it holds that $((Q^l)^{-1}f^l)_1 = (\hat{q}^l)^{-1}(\hat{b}_1^l, \ldots, \hat{b}_p^l)^t$, where $\hat{b}_i^l$ is the coefficient of s^{r-1} of b_i^l and $\hat{q}^l \neq 0$ is the coefficient of s^r in $\det Q^l$. Since $\hat{q}^\infty \neq 0$, the result follows. ∎

With this preparation we can state the continuity of the Fuhrmann realization in the mentioned case. The only thing which remains to be done, is to find continuous bases in K_{Q^l}.

Theorem 3.4. *Let* $[P^l, Q^l] \in \mathbb{R}[s]^{p \times (m+p)}$ *with* $(Q^l)^{-1}P^l$ *proper,* $\deg \det Q^l = r$ *for all* $l \in \overline{\mathbb{N}}$ *and* $[P^l, Q^l] \to [P^\infty, Q^\infty]$. *Then there exist matrices* $(A, B, C, D) \in \Sigma(r, m, p)$ *with* $(A^l, B^l, C^l, D^l) \to (A^\infty, B^\infty, C^\infty, D^\infty)$ *and* $\ker[P^l(\sigma), Q^l(\sigma)] = \mathscr{B}(A^l, B^l, C^l, D^l)$.

Proof. 1. Let $D^l = -(Q^l)^{-1}P^l(\infty)$. Then from Proposition 3.3(a) we get $D^l \to D^\infty$ Moreover, it holds $P^l + Q^l D^l \to P^\infty + Q^\infty D^\infty$. Thus we can restrict to the case $(Q^l)^{-1}P^l$ being strictly proper.

2. From Definition 1.2 follows the existence of $N \in \mathbb{N}$ such that $Q^l(s) = \sum_{i=0}^N Q_i^l s^i$ for all $l \in \overline{\mathbb{N}}$. Then the strict properness of $(Q^l)^{-1}f$ for $f \in K_{Q^l}$ implies $K_{Q^l} \subseteq \mathscr{K} = \{f \in \mathbb{R}[s]^p \mid f(s) = \sum_{i=0}^{N-1} f_i s^i \text{ with } f_i \in \mathbb{R}^p\}$. $\mathscr{K}$ is a vector space over $\mathbb{R}$ with basis $\mathscr{A} = \{s^i e_j \mid i = 0, \ldots, N-1, j = 1, \ldots, p\}$.

3. Let $\mathscr{F}^\infty = \{f_1^\infty, \ldots, f_r^\infty\}$ be a basis of K_{Q^∞} with $f_i^\infty = \Pi_{Q^\infty} g_i$ for some $g_i \in \mathbb{R}[s]^p$. Define $f_i^l = \Pi_{Q^l} g_i \in K_{Q^l}$. Then, by Proposition 3.3(b) it holds that $f_i^l \to f_i^\infty$. This implies that $\mathscr{F}^l = \{f_1^l, \ldots, f_r^l\}$ is a basis of K_{Q^l} for sufficiently large l.

4. Hence we can assume that $\mathscr{F}^l$ as given in part 3 is a basis of K_{Q^l} for all $l \in \overline{\mathbb{N}}$ and $f_\nu^l \to f_\nu^\infty$. Let $f_\nu^l = \sum_{i=0}^{N-1} \sum_{j=1}^p (a_\nu^l)_{ij}(s^j e_i)$ and put $M^l = ((a_\nu^l)_{ij}) \in \mathbb{R}^{pN \times r}$, where $\nu = 1, \ldots, r$ is the column index and $(i, j) \in \{0, \ldots, N-1\} \times \{1, \ldots, p\}$ is the row index. Thus, the νth column of M^l consists of the coefficients of f_ν^l with respect to the basis $\mathscr{A}$ of $\mathscr{K}$. It follows that $\operatorname{rk} M^l = r$ for all $l \in \overline{\mathbb{N}}$ and by assumption $M^l \to M^\infty$.

Let $b_\nu^l \in \mathbb{R}$ for $l \in \overline{\mathbb{N}}$ and $\nu = 1, \ldots, r$ so that $\sum_{\nu=1}^r b_\nu^l f_\nu^l \to \sum_{\nu=1}^r b_\nu^\infty f_\nu^\infty$ (in the sense of Definition 1.2, which means here, in the coefficients with respect to the basis $\mathscr{A}$ of $\mathscr{K}$). This yields for $b^l = (b_1^l, \ldots, b_r^l)^t \in \mathbb{R}^r$ the convergence $M^l b^l \to M^\infty b^\infty$. Since $\operatorname{rk} M^\infty = r$, there exist continuous left inverses for M^l and this implies $b^l \to b^\infty$.

The above, together with Proposition 3.3(b), shows that the operator A^l: $K_{Q^l} \to K_{Q^l}$, $f \mapsto \Pi_{Q^l}(sf)$ has a continuous matrix representation if we choose a basis as in part 3. The same argument yields a continuous matrix representation for B^l: $\mathbb{R}^m \to K_{Q^l}$, $\xi \mapsto -P^l(s)\xi$. Proposition 3.3(b) also shows that C^l: $K_{Q^l} \to \mathbb{R}^p$, $f \mapsto ((Q^l)^{-1}f)_1$ leads to a continuous matrix. ∎

Remark 3.5. (a) Let us look again at Proposition 2.1. Let Q^l, P^l be written as in (2.1), (2.2). Since the construction of the representation of $\ker[P^l(\sigma), Q^l(\sigma)]$ as a singular system is only a reordering of all the coefficients in $[P^l, Q^l]$, this gives easily a continuous representation, if $[P^l, Q^l] \to [P^\infty, Q^\infty]$ and $\hat{P}^l = \hat{Q}^l D^l$ for $l \in \overline{\mathbb{N}}$ with $D^l \to D^\infty$. This second requirement does not follow from $[P^l, Q^l] \to [P^\infty, Q^\infty]$, as one can see from the little example $[P^l, Q^l] = [\hat{P}^l, \hat{Q}^l] = [(-1)^l l^{-1}, l^{-1}]$. But it is easily fulfilled if one chooses the n_i in (2.1), (2.2) large enough so that $[\hat{P}, \hat{Q}] = 0$. Hence we always get continuous *singular* representations. In this context, the question about continuous singular representations, which are minimal (in some sense), or about continuous i/s/o representations, which are allowed to degenerate in the limit to a singular one, seems to be quite interesting and much more difficult.

(b) In the case $\hat{Q}^l \in Gl_p(\mathbb{R})$ for all $l \in \overline{\mathbb{N}}$ we have $-(Q^l)^{-1}P^l(\infty) = D^l \to D^\infty$, hence we get already a continuous i/s/o representation with Proposition 2.1. But this is just the very special case, where $[P^l, Q^l]$ is row-proper, $(Q^l)^{-1}P^l$ is proper for all $l \in \overline{\mathbb{N}}$ and the row-degrees all remain fixed in the limit. Example 3.6 shows that $R^l \to R^\infty$ in general does not imply the existence of $U^l \in Gl_p(\mathbb{R}[s, s^{-1}])$ with $U^l R^l$ being row-proper for all $l \in \mathbb{N}$ and $U^l R^l \to U^\infty R^\infty$.

(c) Since the Fuhrmann realization is a minimal representation if we restrict to time-axis $\mathbb{Z}_+$ or if we consider the continuous-time case (see Remark 2.6), we get for these cases in Theorem 3.4 a converging sequence of *minimal* i/s/o representations.

(d) Let us make a comparison with the continuity results given in [RR2] for the continuous-time case. In [RR1] a smooth compactification of the space of all AR systems is constructed via homogenization. This compact space also contains systems of lower degree. In [RR2] a homeomorphism between this compact space and an orbit space of pencil representations of the type $G\dot{z} = Fz$, $w = Hz$ under similarity $(G, F, H) \sim (SGT^{-1}, SFT^{-1}, HT^{-1})$ is given. This type of representations were extensively studied in [KS]. It is easy to see that these representations generalize i/s/o representations. Following the construction in [RR2] one can prove that Theorem 3.4 gives an explicit version for the homeomorphism in [RR2] in the special situation under consideration. More precisely, let $[P^l, Q^l] \in \mathbb{R}[s]^{p\times(m+p)}$ be as in Theorem 3.4. Then the homogenized matrices do all have the same degree and converge in the sense of [RR2]. Thus, by [RR2] there exists a converging minimal representation (G^l, F^l, H^l) for $[P^l, Q^l]$. The properness of $(Q^l)^{-1}P^l$ implies that the data of (G^l, F^l, H^l) can be reorganized in order to get a converging minimal sequence (A^l, B^l, C^l, D^l) of i/s/o representations.

Example 3.6. For $l \in \overline{\mathbb{N}}$ let $\varepsilon = l^{-1}$ and put

$$R^l = \begin{bmatrix} 1 & \varepsilon s + 1 & 1 \\ 0 & s^2 & \varepsilon \end{bmatrix} \to R^\infty = \begin{bmatrix} 1 & 1 & 1 \\ 0 & s^2 & 0 \end{bmatrix} \in \mathbb{R}[s]^{2\times 3}.$$

Since R^∞ is row-proper with row degrees (0, 2), but R^l has the minimal row indices (1, 1), these matrices cannot be left equivalent to converging *row-proper* matrices $U^l R^l \to U^\infty R^\infty$ with $U^l \in Gl_p(\mathbb{R}[s, s^{-1}])$ for all $l \in \overline{\mathbb{N}}$.

The construction of the Fuhrmann realization leads to the following:

$$K_{Q^l} = \left\{ \begin{pmatrix} \varepsilon a \\ as - b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \in \mathbb{R}[s]^2$$

for all $l \in \overline{\mathbb{N}}$ serves as the abstract state space. With respect to the basis $\{(0, -1)^t, (\varepsilon, s)^t\}$ of K_{Q^l} the operators of Proposition 2.5 lead to the i/s/o representation

$$\sigma x_1 = -x_2, \qquad y_1 = x_1,$$
$$\sigma x_2 = -\varepsilon x_1 + \varepsilon^2 x_2 - \varepsilon u, \qquad y_2 = -x_1 + \varepsilon x_2 - u,$$

which holds in fact for all $l \in \overline{\mathbb{N}}$. Notice that the McMillan degree (see [W3, p. 276]) of R^l is equal to 2 if $l \in \mathbb{N}$ and equal to 0 if $l = \infty$; see Remark 2.6 and note that rk $R^\infty(0) < 2$.

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