A convolution algebra of delay-differential operators and a related problem of finite spectrum assignability

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Abstract: In this paper we investigate a ring of delay-differential operators, which appears in the study of time-delay systems with commensurate point-delays. Besides point-delay differential operators, this ring contains also certain distributed delays. It can be described as a certain algebraic extension of the ring of point-delay differential operators as well as a convolution algebra of compact support distributions. It will be shown that the finite spectrum assignability problem can be solved for spectrally controllable systems with controllers from this algebra.

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1 Introduction

Time-delay systems, even of the simplest type including only finitely many commensurate point-delays, are infinite-dimensional systems and may therefore be treated best with (functional) analytic methods. The possibility of performing nevertheless an algebraic theory for these specific systems relies on some finiteness features. Apparently it was first in [M], where a polynomial ring in several indeterminates was proposed as a model for time-delay systems. In the case of commensurate point-delays this model looks as follows. Denote by σ the forward-shift of unit length, i. e. $\sigma w(t) := w(t-1)$ for a function $w : \mathbb{R} \to \mathbb{R}^m$. Then each polynomial $p = \sum_{j=0}^{L} \sum_{i=0}^{N} p_{ij} s^i z^j \in \mathbb{R}[s,z]$ in two variables gives rise to the delay-differential operator

$$\big(p(\tfrac{d}{dt},\sigma)w\big)(t) = \Big(\big(\sum_{j=0}^{L}\sum_{i=0}^{N} p_{ij}\frac{d^i}{dt^i}\circ\sigma^j\big)w\Big)(t) = \sum_{j=0}^{L}\sum_{i=0}^{N} p_{ij} w^{(i)}(t-j). \tag{1.1}$$

In this paper we will adopt this model for delay-differential equations on $\mathcal{C}^\infty := \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, the space of real-valued infinitely differentiable functions defined on $\mathbb{R}$. The restriction to $\mathcal{C}^\infty$ is basically only for the sake of simplicity. Broader classes of functions are possible within this framework, as we will point out in Rem. 2.9. However, for $w \in \mathcal{C}^\infty$ equation (1.1) makes sense for all $t \in \mathbb{R}$ and we can even allow backward-shifts σ^{-j}. That means, we have the embedding

$$\begin{array}{ccc} \mathbb{R}[s,z,z^{-1}] & \longrightarrow & \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty) \\ p = \sum_{j=l}^{L}\sum_{i=0}^{N} p_{ij}s^i z^j & \longmapsto & p(\frac{d}{dt},\sigma) = \sum_{j=l}^{L}\sum_{i=0}^{N} p_{ij}\frac{d^i}{dt^i}\circ\sigma^j \end{array} \tag{1.2}$$

In this set-up the kernel of $p(\frac{d}{dt},\sigma)$ is given as the space of all $\mathcal{C}^\infty$-solutions of the homogeneous delay-differential equation $p(\frac{d}{dt},\sigma)w = 0$, i. e.

$$\ker p(\tfrac{d}{dt},\sigma) := \Big\{ w \in \mathcal{C}^\infty \,\Big|\, \sum_{j=l}^{L}\sum_{i=0}^{N} p_{ij} w^{(i)}(t-j) = 0\ \forall\, t \in \mathbb{R} \Big\}. \tag{1.3}$$

The paper [M] can be regarded as the starting point of the interpretation of a time-delay system as a system over a ring, see also the survey [So]. Similarly to [M], in [K1] a polynomial ring, this time with Dirac-distributions as the indeterminates, is introduced as a convolution algebra operating on the Schwartz-space of distributions having support bounded to the left. With this algebra a theory for the internal description of input-output equations is developed in [K1].

To our knowledge, in the more ring theoretical literature, a wider class of delay operators, including distributed delays, was first presented in [K2]. A ring of distributions, consisting of L_+^{loc}-functions and formal sums of Dirac-impulses, is used to incorporate also the initial data in the framework, so that solutions to the initial value problem can be obtained via an appropriate operational calculus. In [KKT2] a ring Θ generated by the entire functions $\theta_\lambda(s) = \frac{1-e^{-s}e^{\lambda}}{s-\lambda}$ and their derivatives is introduced in order to achieve Bézout-identities $(sI - A(e^{-s}))M(s) + B(e^{-s})N(s) = I$ with coefficient matrices over the extension $\Theta[s,e^{-s}]$. In a step further, this was exploited to obtain proper stable Bézout-factorizations for transfer functions of systems with commensurate delays. The generating function θ_λ is known to be the transfer function of a specific distributed delay. This approach was recently resumed in [BL2]. With a slight modification of the ring Θ the authors are able to prove that $\Theta[s,e^{-s}]$ is a Bézout-domain. They use this fact to solve a specific finite spectrum assignment problem for the systems under consideration. This is quite parallel to the last section of the paper at hand. In fact, with a different approach in [BL1] one can show, that $\Theta[s,e^{-s},e^{s}]$ is isomorphic to the ring

$$\mathcal{H} := \Big\{ \frac{p}{r} \,\Big|\, p,\, r \in \mathbb{R}[s,z],\, r \neq 0,\, \ker r(\tfrac{d}{dt},\sigma) \subseteq \ker p(\tfrac{d}{dt},\sigma) \Big\}, \tag{1.4}$$

which will be the starting point for our investigation of this algebra. The fact that $\mathcal{H}$ is indeed a ring and can be regarded as an operator algebra will be made precise in the next section. With this definition we resume the algebraic approach of [G1]. Therein, after a thorough study of the algebraic properties of $\mathcal{H}$, the ring has been exploited for the study of the uniqueness of kernel-representations for behaviors given by delay-differential equations. In a very different context, the ring of Laplace transforms of $\mathcal{H}$ has been used in the paper [OP] to show the coincidence of null controllability and spectral controllability for the class of systems under consideration.

The above cited papers should indicate that, from a system theoretical point of view, this larger ring of operators rather than the polynomial ring $\mathbb{R}[\frac{d}{dt},\sigma]$ appears to be an adequate place for the solution of specific control problems, if not even necessary like for finite spectrum assignment. The fact, that some of these problems turn out to be easily solvable (at least theoretically) within $\mathcal{H}$, is based on some nice algebraic property of this ring. Indeed, $\mathcal{H}$ is a Bézout-domain [G1], a fact, which is often much more useful than the Noetherian property for polynomial rings.

In this paper we will give various equivalent characterizations for the ring $\mathcal{H}$, thereby rediscovering parts of the existing results given in the papers cited above and combining them into one picture. The basis for this approach is the definition of $\mathcal{H}$ in (1.4) as well as its Bézout-property. All assertions will be derived from these facts. The main result of Section 2 is that $\mathcal{H}$ is a certain convolution algebra of compact support distributions. Although this might look somehow obvious using Laplace-transforms, we think it is worth seeing how it can be proven directly from (1.4). However, the importance of this result within the context of this paper rests on the fact, that it allows to abandon the restriction to $\mathcal{C}^\infty$-functions as used in (1.3) and (1.4). From an algebraic point of view the choice of $\mathcal{C}^\infty$ is very convenient to begin with, simply because it is a module over $\mathbb{R}[\frac{d}{dt}, \sigma, \sigma^{-1}]$. Over the proper part of $\mathcal{H}$, however, much more general spaces like $L_1(\mathbb{R})$ are modules with respect to convolution, too.
In the third section we consider the question of finite spectrum assignability for time-delay systems. The solution to this problem, or rather a slight generalization, will be derived via a Bézout-equation over $\mathcal{H}$. Using the results of Section 2, it will be shown how the corresponding controller looks like. This way we rediscover certain results from the existing literature and see how they fit into this algebraic framework.

It is worth mentioning that the approach does not carry over to the more general case of differential equations with non-commensurate delays. It has been shown in [H2] that in this case the corresponding operator ring $\mathcal{H}$ is not a Bézout-domain. An essentially algebraic theory for delay-differential systems with non-commensurate delays has been developed recently in [V].

The paper follows the behavioral approach to linear systems, meaning basically that a system is not defined through its transfer function but rather via the space of all solutions satisfying certain given sets of equations. This is only a matter of choice, it does not really affect the results of this paper.

2 A ring of distributed delay operators

We start off with the definition of $\mathcal{H}$ in (1.4). Several equivalent descriptions of this space will be derived throughout this section. First of all, from the very definition it is obvious, that $\mathcal{H}$ can be embedded into $\mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty)$ via

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty) \\ q := \dfrac{p}{r} & \longmapsto & \widetilde{q} := p(\frac{d}{dt}, \sigma) \circ r(\frac{d}{dt}, \sigma)^{-1} \end{array} \tag{2.1}$$

For the well-definedness see [G1, 2.8]. More precisely, for $w \in \mathcal{C}^\infty$ it is $\widetilde{q}w = p(\frac{d}{dt}, \sigma)v$, where $v \in \mathcal{C}^\infty$ satisfies $r(\frac{d}{dt}, \sigma)v = w$; for the surjectivity of $r(\frac{d}{dt}, \sigma)$ see also [G1, 2.10]. In the following we will use the symbol $\widetilde{p}$ instead of $p(\frac{d}{dt}, \sigma)$ also for the delay-differential operators in $\mathbb{R}[\frac{d}{dt}, \sigma, \sigma^{-1}]$.

The definition of $\mathcal{H}$ in (1.4) is only fruitful, if there is an alternative way to see if a quotient $\frac{p}{r}$ is in $\mathcal{H}$, hence if the inclusion $\ker \widetilde{r} \subseteq \ker \widetilde{p}$ holds true. In order to do so, the following notations will be useful.

Definition 2.1 *Let $H(\mathbb{C})$ be the ring of entire functions. For $q = pr^{-1} \in \mathbb{R}(s, z)$ with $p, r \in \mathbb{R}[s, z]$ denote by q^* the meromorphic function given by $q^*(s) = q(s, e^{-s})$ defined on $\{\lambda \in \mathbb{C} \mid r(\lambda, e^{-\lambda}) \neq 0\}$.*

Exploiting the characteristic functions associated with $\mathbb{R}[\frac{d}{dt},\sigma,\sigma^{-1}]$ the following has been established in [G1, 2.7,3.1,4.1].

Theorem 2.2 *The set $\mathcal{H}$ is a subring of $\mathbb{R}(s,z)$ which can be characterized in the following ways*

$$\mathcal{H}=\{q\in\mathbb{R}(s,z)\,|\,q^*\in H(\mathbb{C})\}=\{q\in\mathbb{R}(s)[z,z^{-1}]\,|\,q^*\in H(\mathbb{C})\}.$$

The bijectivity of σ immediately implies $\ker(p(\frac{d}{dt},\sigma)\circ\sigma^k)=\ker p(\frac{d}{dt},\sigma)$, indicating that also polynomials in z^{-1} appear naturally in this framework. The above theorem says that only expressions of the form $\phi(s)z^k$ with $\phi\in\mathbb{R}[s]$ can occur in the denominators of the elements of $\mathcal{H}$.

By its very definition, it is immediate that $\mathcal{H}$ is the largest subring of $\mathbb{R}(s,z)$ containing $\mathbb{R}[s,z,z^{-1}]$ to which the embedding (1.2) can be extended.

Example 2.3 Consider $q=\frac{e^{\lambda L}z^L-1}{s-\lambda}\in\mathcal{H}$ with $\lambda\in\mathbb{R}$ and $L\in\mathbb{N}$. Note that for $L=1$ it is $q^*(s)=-\theta_\lambda(s)$ from the introduction. In order to calculate $\widetilde{q}w$ for $w\in\mathcal{C}^\infty$, we need first to find $v\in\mathcal{C}^\infty$ fulfilling $(\frac{d}{dt}-\lambda)v=w$. With $v(t)=\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau$ we then obtain

$$\widetilde{q}w(t)=(e^{\lambda L}\sigma^L-1)v(t)=-\int_0^L e^{\lambda\tau}w(t-\tau)d\tau,$$

which is known as a distributed delay. In the Thms. 2.8 and 2.10 it will be described precisely which distributed delay operators originate from $\mathcal{H}$.

In the last section about weak coefficient assignability we will need the following subrings of $\mathcal{H}$. Define

$$\mathcal{H}_0:=\mathcal{H}\cap\mathbb{R}(s)[z]=\Big\{\frac{p}{\phi}\,\Big|\,p\in\mathbb{R}[s,z],\ \phi\in\mathbb{R}[s]\backslash\{0\},\ \frac{p^*}{\phi}\in H(\mathbb{C})\Big\},\tag{2.2}$$

$$\mathcal{H}_{0,\mathsf{sp}}:=\Big\{\frac{p}{\phi}\in\mathcal{H}_0\,\Big|\,\deg_{\mathrm{s}}p<\deg\phi\Big\}\subset\mathcal{H}_{0,\mathsf{p}}:=\Big\{\frac{p}{\phi}\in\mathcal{H}_0\,\Big|\,\deg_{\mathrm{s}}p\le\deg\phi\Big\}\subset\mathcal{H}_0,\tag{2.3}$$

where $\deg_{\mathrm{s}}p$ denotes the degree of $p\in\mathbb{R}[s,z]$ with respect to s. In these subrings no polynomials in z^{-1} are allowed. Using $\mathcal{H}_0$ instead of $\mathcal{H}$ one avoids advanced delay-differential equations containing the backward shift σ^{-1}. The operators from $\mathcal{H}_{0,\mathsf{sp}}$ will shown to be convolution operators with certain piecewise continuous functions, as it was the case in Exp. 2.3.

The rings $\mathcal{H}_0$ and $\mathcal{H}_{0,\mathsf{sp}}$ occured already in a different context in [OP] as $R_{\mathrm{E}}(s)$ and $R_{\mathrm{SPE}}(s)$ respectively. Also, in [KKT2] the ring $\mathcal{H}_0$ was introduced, but in a completely different way. It turns out that $\mathcal{H}_0$ is the ring $\Theta[z,s]$, while $\mathcal{H}_{0,\mathsf{sp}}$ is identical with $\Theta[z]$, both given at [KKT2, p. 841]. In [BL2] and [BL1] the approach of [KKT2] has been resumed.

From the very definition one obtains immediately the decomposition

$$\mathcal{H}_0=\mathcal{H}_{0,\mathsf{sp}}\oplus\mathbb{R}[s,z].\tag{2.4}$$

In fact, if $q=\frac{p}{\phi}\in\mathcal{H}_0$ with $\deg_{\mathrm{s}}p\ge\deg\phi$, then $p=a\phi+b$ with $a,\,b\in\mathbb{R}[s,z]$ and $\deg_{\mathrm{s}}b<\deg\phi$, thus

$$q=\frac{b}{\phi}+a\in\mathcal{H}_{0,\mathsf{sp}}+\mathbb{R}[s,z].\tag{2.5}$$

The operator rings $\mathcal{H}$ and $\mathcal{H}_0$ have some algebraic advantages compared to $\mathbb{R}[s,z]$ or $\mathbb{R}[s,z,z^{-1}]$. In [G1] the following theorem about the ring $\mathcal{H}$ is established. A short glance at the proofs of [G1, 3.1, 3.2] ensures that the same reasoning is true for $\mathcal{H}_0$.

Theorem 2.4 [G1, 3.2] *The rings $\mathcal{H}$ and $\mathcal{H}_0$ are Bézout-domains, i. e. every finitely generated ideal is principal.*

This and an even nicer property imply triangular and diagonal forms for matrices over these rings, which in turn allows a study of systems of delay-differential equations of the form $\widetilde{P}w = 0$ with $P \in \mathcal{H}^{r\times n}$ and $w \in (\mathcal{C}^\infty)^n$ in the behavioral framework [G1].

In certain cases, basically with one of the factors being monic in s, a Bézout-identity has already been derived before, see [OP, Sect. 4], [KKT2, (3.2), (4,14)], and also Rem. 3.3 i) of this paper. In [H2] the corresponding ring for time-delay systems with non-commensurate point-delays has been investigated. It turns out that in that case the ring is not Bézout [H2, 5.13].

Remark 2.5 For practical purposes it is important to be able to solve Bézout-equations in $\mathcal{H}_0$ (resp. $\mathcal{H}$) in a constructive way. That means, for given $p_1, \dots, p_n \in \mathcal{H}_0$ one has to find a greatest common divisor $d \in \mathcal{H}_0$ and coefficients $a_i \in \mathcal{H}_0$ satisfying $d = \sum_{i=1}^n a_i p_i$. We will outline a procedure for the case, where $p_1, \dots, p_n \in \mathcal{H}_0$ are coprime, hence $d = 1$. It can easily be adjusted to the general case. The given procedure is a slight generalization of [BL2, Thm. 1].
Step 1: Since $p_1, \dots, p_n$ are also coprime as elements of $\mathbb{R}(s)[z]$, one can use Euclid's algorithm to find $\hat{a}_1, \dots, \hat{a}_n \in \mathbb{R}(s)[z]$ such that $1 = \sum_{i=1}^n \hat{a}_i p_i$. This can be rewritten as

$$\phi = \sum_{i=1}^n a_i p_i \text{ for some } \phi \in \mathbb{R}[s] \text{ and } a_i \in \mathcal{H}_0. \tag{2.6}$$

Actually, at this point $a_i \in \mathbb{R}[s, z]$, but throughout the upcoming procedure the coefficients will change to elements of $\mathcal{H}_0$.
Step 2: We have to adjust the coefficients a_i so that the zeros of ϕ can be removed. Assume $\phi(\lambda) = 0$ for some $\lambda \in \mathbb{C}$. The coprimeness of $p_1, \dots, p_n$ in $\mathcal{H}_0$ is equivalent to the coprimeness of $p_1^*, \dots, p_n^*$ in $H(\mathbb{C})$, see [G1, 3.1]. Therefore there is at least one non-zero $p_i^*(\lambda)$, say $p_1^*(\lambda) \neq 0$. Then (2.6) implies

$$\begin{pmatrix} a_1^*(\lambda) \\ a_2^*(\lambda) \\ \vdots \\ a_n^*(\lambda) \end{pmatrix} \in \ker_{\mathbb{C}}[p_1^*(\lambda), \dots, p_n^*(\lambda)] = \operatorname{im}_{\mathbb{C}} \begin{bmatrix} p_2^*(\lambda) & \cdots & p_n^*(\lambda) \\ -p_1^*(\lambda) & & \\ & \ddots & \\ & & -p_1^*(\lambda) \end{bmatrix}$$

with all non-specified entries of the matrix being zero. Let

$$\begin{pmatrix} a_1^*(\lambda) \\ a_2^*(\lambda) \\ \vdots \\ a_n^*(\lambda) \end{pmatrix} = \begin{bmatrix} p_2^*(\lambda) & \cdots & p_n^*(\lambda) \\ -p_1^*(\lambda) & & \\ & \ddots & \\ & & -p_1^*(\lambda) \end{bmatrix} \begin{pmatrix} c_2 \\ \vdots \\ c_n \end{pmatrix}$$

with $c_2, \dots, c_n \in \mathbb{C}$. Choose polynomials $\gamma_2, \dots, \gamma_n \in \mathbb{R}[s]$ such that $\gamma_i(\lambda) = c_i$. If λ is real, then the c_i are real, too, and γ_i is simply a constant, otherwise γ_i is at most quadratic. Now, if $\lambda \in \mathbb{C}$, (2.6) can be rewritten as

$$\frac{\phi}{(s-\lambda)(s-\bar{\lambda})} = \frac{a_1 - \gamma_2 p_2 - \ldots - \gamma_n p_n}{(s-\lambda)(s-\bar{\lambda})} p_1 + \sum_{i=2}^n \frac{a_i + \gamma_i p_1}{(s-\lambda)(s-\bar{\lambda})} p_i$$

where all coefficients are indeed in $\mathcal{H}_0$ and the left-hand side is a polynomial of lower degree. The real case looks analogous.

In this way we can repeat Step 2 until ϕ is a constant.
A special case is worth mentioning. If one of the p_i is in $\mathbb{R}[s]$, say p_n, then (2.6) holds true with $\phi = p_n$, $a_n = 1$, $a_i = 0$ for $i < n$. A careful study of the above procedure shows that one can always choose $\gamma_i = 0$ for $i \neq n$, which then leads to a Bézout-equation $1 = \sum_{i=1}^{n} a_i p_i$ with $a_1, \dots, a_{n-1} \in \mathbb{R}[s]$ and $a_n \in \mathcal{H}_0$. $\square$

In the second part of this section we are going to describe $\mathcal{H}$ and $\mathcal{H}_0$ as certain convolution algebras of compact support distributions.

Let $\mathcal{D}'$ be the $\mathbb{R}$-vector-space of distributions on $\mathbb{R}$ with $\mathcal{D} := \{f \in \mathcal{C}^\infty \mid \operatorname{supp} f \text{ is compact}\}$ as the space of test-functions. Here $\operatorname{supp} f$ denotes the support of the function f. Furthermore, let

$$\mathcal{D}'_+ := \{T \in \mathcal{D}' \mid \operatorname{supp} T \text{ bounded from the left}\} \quad\text{and}\quad \mathcal{D}'_{\mathsf{c}} := \{T \in \mathcal{D}' \mid \operatorname{supp} T \ \text{compact}\}.$$

Denote with $\delta_a^{(k)}$ the k-th derivative of the Dirac-distribution at $a \in \mathbb{R}$. Recall that the convolution $S * T$ of distributions is well-defined and commutative if either both factors are in $\mathcal{D}'_+$ or if at least one factor is in $\mathcal{D}'_{\mathsf{c}}$. Moreover, it is associative on $\mathcal{D}'_+$ or if at least two of the three factors are in $\mathcal{D}'_{\mathsf{c}}$. Even more, $(\mathcal{D}'_+, +, *)$ is a domain with δ_0 as the identity. For details see [S2, p. 14, p. 28/29] or [Z, p. 124-129].

In this set-up differentiation (resp. forward-shift) corresponds to convolution with $\delta_0^{(1)}$ (resp. δ_1). Hence for $p = \sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} s^i z^j \in \mathbb{R}[s, z, z^{-1}]$ and $w \in \mathcal{C}^\infty$ we have

$$\widetilde{p}w = \Big(\sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} \delta_0^{(i)} * \delta_j\Big) * w = p(\delta_0^{(1)}, \delta_1) * w \quad \text{for all } w \in \mathcal{C}^\infty \tag{2.7}$$

and $\mathbb{R}[\delta_0^{(1)}, \delta_1, \delta_{-1}]$ is a subring of $\mathcal{D}'_+$ isomorphic to $\mathbb{R}[s, z, z^{-1}]$.

For the subsequent discussions we will also need the function space

$$\mathcal{PC}^\infty := \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l} \exists\, t_k \in \mathbb{R} \text{ for } k \in \mathbb{Z} \text{ with } t_k < t_{k+1},\ \lim_{k \to \pm\infty} t_k = \pm\infty, \\ f|_{(t_k, t_{k+1}]} \in \mathcal{C}^\infty((t_k, t_{k+1}], \mathbb{R}) \text{ and } f(t_k+) \in \mathbb{R} \end{array} \right\} \tag{2.8}$$

of piecewise smooth functions, which are left-smooth everywhere and bounded on every finite interval. Note that $\mathcal{C}^\infty \subset \mathcal{PC}^\infty \subset \mathcal{D}'$. Let

$$\mathcal{PC}^\infty_+ := \{f \in \mathcal{PC}^\infty \mid \operatorname{supp} f \text{ bounded from the left}\} \subset \mathcal{D}'_+.$$

By use of the left-derivatives we can extend the delay-differential operators $\widetilde{p}$ for $p \in \mathbb{R}[s, z, z^{-1}]$ from $\mathcal{C}^\infty$ to $\mathcal{PC}^\infty$. Observe, that for $w \in \mathcal{PC}^\infty$ equation (2.7) does not hold true anymore, as can be readily verified by choice of $p = s$ and w being the Heaviside-function. Instead, for a piecewise smooth function f as in (2.8) the following formula is valid [S1, p. 37/38].

$$\delta_j^{(i)} * f = \sigma^j f^{(i)} + \sum_{\mu=0}^{i-1} \sum_{k \in \mathbb{Z}} \Big(f^{(i-1-\mu)}(t_k+) - f^{(i-1-\mu)}(t_k)\Big) \delta_{t_k+j}^{(\mu)}, \tag{2.9}$$

where the sum vanishes if $i = 0$. Note that $\sigma^j f^{(i)} \in \mathcal{PC}^\infty$.

The next theorem is the main step towards an interpretation of $\mathcal{H}$ as a space of distributions. It has been established in [K1, Prop. 4]. However, we want to present a short sketch of the proof in our notation as we need parts of it in the upcoming discussion.

Theorem 2.6 *$\mathbb{R}(\delta_0^{(1)}, \delta_1)$ is a subfield of $\mathcal{D}'_+$.*

PROOF: The inclusion $\mathbb{R}(\delta_0^{(1)}) \subset \mathcal{D}'_+$ is a standard result in distribution theory, see e. g. [Z, 6.3-1]. Indeed, the inverse of $\phi(\delta_0^{(1)}) = \sum_{i=0}^r \phi_i \delta_0^{(i)}$ in $\mathcal{D}'_+$ for a given polynomial $\phi = \sum_{i=0}^r \phi_i s^i \in \mathbb{R}[s]$ exists already in the subspace $\mathcal{PC}_+^\infty \subset \mathcal{D}'_+$ and is given by $f \in \mathcal{PC}_+^\infty$ defined as

$$f(t) = \begin{cases} 0 & \text{for } t \le 0 \\ h(t) & \text{for } t > 0 \end{cases} \text{ with } h \in \ker \widetilde{\phi} \subset \mathcal{C}^\infty \text{ and } h^{(i)}(0) = \begin{cases} 0 & \text{for } i = 0, \dots, r-2 \\ \phi_r^{-1} & \text{for } i = r-1 \end{cases} \tag{2.10}$$

The inverse of $p(\delta_0^{(1)}, \delta_1)$ in $\mathcal{D}'_+$ where $p \in \mathbb{R}[s,z]\backslash\{0\}$ can be obtained by first calculating the inverse $p^{-1} = \sum_{j=l}^\infty p_j(s) z^j \in \mathbb{R}(s)((z))$ as a formal Laurent series in z. By i) the distributions $p_j(\delta_0^{(1)}) * \delta_j$ exist in $\mathcal{D}'_+$. Since they have support in $[j, \infty)$, the series $\sum_{j=l}^\infty p_j(\delta_0^{(1)}) * \delta_j$ converges in $\mathcal{D}'_+$ and by continuity of the convolution is equal to $p(\delta_0^{(1)}, \delta_1)^{-1}$. □

Example 2.7 Let us compute $q(\delta_0^{(1)}, \delta_1) \in \mathbb{R}(\delta_0^{(1)}, \delta_1)$ for $q = \frac{e^{\lambda L} z^L - 1}{s - \lambda}$. According to part i) of the above proof define $f \in \mathcal{PC}_+^\infty$ as $f(t) = 0$ for $t \le 0$ and $f(t) = e^{\lambda t}$ if $t > 0$. Then $q(\delta_0^{(1)}, \delta_1) = (e^{\lambda L}\delta_L - 1) * f = e^{\lambda L}\sigma^L f - f =: g \in \mathcal{PC}_+^\infty$ with $g(t) = -e^{\lambda t}$ for $t \in (0, L]$ and $g(t) = 0$ elsewhere. Observe that g has compact support and therefore defines a convolution operator on $\mathcal{C}^\infty$

$$(g * w)(t) = \int_{\mathbb{R}} g(\tau) w(t-\tau) d\tau = -\int_0^L e^{\lambda\tau} w(t-\tau) d\tau \text{ for } w \in \mathcal{C}^\infty,$$

which is exactly $\widetilde{q}w$ as calculated in Exp. 2.3.

Since $\mathcal{H} \subseteq \mathbb{R}(s)[z, z^{-1}]$, we need the following explicit expression for distributions in $\mathbb{R}(\delta_0^{(1)})[\delta_1, \delta_{-1}]$. Let

$$q = \frac{p}{\phi} \in \mathbb{R}(s)[z, z^{-1}] \text{ with } p = \sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} s^i z^j \in \mathbb{R}[s, z, z^{-1}] \text{ and } \phi = \sum_{i=0}^{r} \phi_i s^i \in \mathbb{R}[s]\backslash\{0\}. \tag{2.11}$$

We can assume $r = \deg \phi > 0$. Let $\phi(\delta_0^{(1)})^{-1} = f$ be as in (2.10). Using (2.9) one obtains

$$q(\delta_0^{(1)}, \delta_1) = \Big(\sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} \delta_j^{(i)}\Big) * f = \widetilde{p}(f) + \sum_{j=l}^{L} \sum_{i=1}^{N} \sum_{\mu=0}^{i-1} p_{ij} f^{(i-1-\mu)}(0+) \delta_j^{(\mu)}, \tag{2.12}$$

where $\widetilde{p}(f)$ refers again to the left-derivative of f. Note that (2.12) is a decomposition of $q(\delta_0^{(1)}, \delta_1)$ into a regular distribution and an impulsive part.

With the next theorem we will see that this decomposition actually reflects (2.4). Precisely, $\mathcal{H}$ is shown to be the set of all compact support distributions inside $\mathbb{R}(\delta_0^{(1)}, \delta_1)$ and the operator $\widetilde{q}$ defined in (2.1) is simply convolution with $q(\delta_0^{(1)}, \delta_1)$, as it was the case in Exp. 2.7. Furthermore, strictly proper functions lead to regular distributions. Parts of the theorem could be proven easily via the Laplace-transform. However, since we are not aware of complete suitable references for each of the statements given below, we give a direct proof based on our approach.

Theorem 2.8

i) The embedding $\mathbb{R}(s,z) \to \mathcal{D}'_+$*,* $q \mapsto q(\delta_0^{(1)}, \delta_1)$ *yields the ring-isomorphisms*

$\mathcal{H} \cong \mathbb{R}(\delta_0^{(1)}, \delta_1) \cap \mathcal{D}'_{\mathsf{c}}$,

$\mathcal{H}_0 \cong \mathbb{R}(\delta_0^{(1)}, \delta_1) \cap \{T \in \mathcal{D}'_{\mathsf{c}} \,|\, \operatorname{supp} T \subset [0,\infty)\}$,

$\mathcal{H}_{0,\mathsf{sp}} \cong \mathbb{R}(\delta_0^{(1)}, \delta_1) \cap \{f \in \mathcal{PC}^\infty \,|\, \operatorname{supp} f \subset [0,\infty)$ *and compact*$\}$.

ii) $q(\delta_0^{(1)}, \delta_1) * w = \widetilde{q}(w)$ *for all* $q \in \mathcal{H}$ *and* $w \in \mathcal{C}^\infty$.

iii) For $q \in \mathcal{H}$ *the Laplace-transform of* $q(\delta_0^{(1)}, \delta_1)$ *is given by* $\mathcal{L}(q(\delta_0^{(1)}, \delta_1)) = q^* \in H(\mathbb{C})$.

Proof: i) We will use the representation (2.12) for all three isomorphisms. Hence let $q \in \mathcal{H}$ be as in (2.11). We have to show that $\widetilde{p}(f)$ has compact support. Since $\phi(\delta_0^{(1)})^{-1} = f$ where $f \in \mathcal{PC}_+^\infty$ is as in (2.10), one obtains $\widetilde{p}(f)(t) = \widetilde{p}(h)(t) = 0$ for $t > L$ as a consequence of $h \in \ker \widetilde{\phi} \subseteq \ker \widetilde{p}$, cf. (1.4). Obviously, $\widetilde{p}(f)(t) = 0$ for $t < l$ and therefore $\operatorname{supp} q(\delta_0^{(1)}, \delta_1)$ is compact. In order to fully establish $\mathcal{H} \cong \mathbb{R}(\delta_0^{(1)}, \delta_1) \cap \mathcal{D}'_{\mathsf{c}}$, it remains to prove that for $q = ab^{-1}$ with $a,\, b \in \mathbb{R}[s,z]$ compactness of $\operatorname{supp} q(\delta_0^{(1)}, \delta_1)$ implies $q \in \mathcal{H}$. By use of (1.4) we need to show $\ker \widetilde{b} \subseteq \ker \widetilde{a}$. To do so, fix $w \in \ker \widetilde{b} \subset \mathcal{C}^\infty$. Using (2.7) and the compactness of $q(\delta_0^{(1)}, \delta_1)$ and $b(\delta_0^{(1)}, \delta_1)$, which ensures the associativity of convolution, one calculates $\widetilde{a}(w) = a(\delta_0^{(1)}, \delta_1) * w = (q(\delta_0^{(1)}, \delta_1) * b(\delta_0^{(1)}, \delta_1)) * w = q(\delta_0^{(1)}, \delta_1) * (b(\delta_0^{(1)}, \delta_1) * w) = 0$, thus $q \in \mathcal{H}$.

For the second isomorphism, note first that for $q \in \mathcal{H}_0$ it is $l = 0$, hence $\operatorname{supp} q(\delta_0^{(1)}, \delta_1) \subset [0,\infty)$. In order to prove that $\operatorname{supp} q(\delta_0^{(1)}, \delta_1) \subset [0,\infty)$ implies $q \in \mathcal{H}_0$, consider again (2.11). We have to show $l \geq 0$. Denote $K := \deg p_l$, where $p_l = \sum_{i=0}^N p_{il} s^i \in \mathbb{R}[s]$. In the case $K \geq r = \deg \phi$, the coefficient of $\delta_l^{(K-r)}$ in (2.12) is given by $p_{Kl} f^{(r-1)}(0+) \neq 0$. Thus the condition on the support implies $l \geq 0$. If $K < r$, the property $h^{(i)}(0) = 0$ for $i = 0, \dots, r-2$ in (2.10) shows that the nonzero function h cannot be in the solution space of $\widetilde{p}_l$. Thus, in this case we obtain for $t \in (l, l+1)$

$$\widetilde{p}(f)(t) = \sum_{j=l}^{L} \sum_{i=0}^{N} p_{ij} f^{(i)}(t-j) = \sum_{i=0}^{K} p_{il} h^{(i)}(t-l) \not\equiv 0$$

which again leads to $l \geq 0$.

For the last isomorphism it only remains to observe that $q \in \mathcal{H}_{0,\mathsf{sp}}$ implies $N < r$ in (2.11), therefore the impulsive part in (2.12) vanishes by construction of f in (2.10).

ii) Let $q = p\phi^{-1} \in \mathcal{H}$. Choose $v \in \mathcal{C}^\infty$ with $\widetilde{\phi}(v) = w$, so that $\widetilde{q}(w) = \widetilde{p}(v)$ by (2.1). Use of (2.7) and the compactness of $\operatorname{supp}\big(p(\delta_0^{(1)}, \delta_1) * \phi(\delta_0^{(1)})^{-1}\big)$, which guarantees associativity in each of the following steps, leads to

$$\begin{aligned} q(\delta_0^{(1)}, \delta_1) * w &= \big(p(\delta_0^{(1)}, \delta_1) * \phi(\delta_0^{(1)})^{-1}\big) * \big(\phi(\delta_0^{(1)}) * v\big) \\ &= \Big(\big(p(\delta_0^{(1)}, \delta_1) * \phi(\delta_0^{(1)})^{-1}\big) * \phi(\delta_0^{(1)})\Big) * v = p(\delta_0^{(1)}, \delta_1) * v = \widetilde{p}(v) = \widetilde{q}(w). \end{aligned}$$

iii) is a direct consequence of the linearity and multiplicativity of $\mathcal{L}$ and of $\mathcal{L}(\delta_j^{(i)}) = s^i e^{-js}$. □

Remark 2.9 The above reasoning shows that for $q \in \mathcal{H}_0$ the decomposition (2.12) corresponds exactly to the decomposition (2.5). The strict proper part of q yields a regular distribution $g := \widetilde{p}(f) \in \mathcal{PC}_+^\infty$, while the polynomial part results in the impulsive sum

$$\sum_{j=0}^{L} \sum_{\mu=0}^{N-1} a_{j\mu} \delta_j^{(\mu)} := \sum_{j=0}^{L} \sum_{i=1}^{N} \sum_{\mu=0}^{i-1} p_{ij} f^{(i-1-\mu)}(0+) \delta_j^{(\mu)}.$$

Hence for $w \in \mathcal{C}^\infty$ it is

$$q(\delta_0^{(1)}, \delta_1) * w = \int_0^L g(\tau) w(\cdot - \tau) d\tau + \sum_{j=0}^{L} \sum_{\mu=0}^{N-1} a_{j\mu} \sigma^j w^{(\mu)}.$$

Moreover, if $q \in \mathcal{H}_{0,\mathsf{sp}}$ the sum vanishes and $\widetilde{q}(w) = q(\delta_0^{(1)}, \delta_1) * w = \int_0^L g(\tau) w(\cdot - \tau) d\tau$ is a simple convolution operator which can be applied to much more general functions than $\mathcal{C}^\infty$. For instance, the spaces $L_1(\mathbb{R})$, $L_1^{\mathsf{loc}}(\mathbb{R})$, $\mathcal{C}(\mathbb{R})$ or $\mathcal{PC}^\infty$ as well as their one-sided versions are all $\mathcal{H}_{0,\mathsf{sp}}$-modules (even $\mathcal{H}_{0,\mathsf{p}}$-modules) with respect to convolution. Without any need for distributions they could be used as underlying function spaces for delay-differential equations within this framework as long as no differentiation occurs. This is e. g. the case for the controller used in Section 3 for weak coefficient assignment, see (3.2) and (3.4). If differentiation is involved, the above function spaces might still be suitable if one generalizes the set-up to weak solutions.

In the last part of this section we will give a third characterization of the operators in $\mathcal{H}_0$, thereby exhibiting again the close relation between $\mathcal{H}_0$ and point-delay systems. It will be shown that the operators from $\mathcal{H}_{0,\mathsf{p}}$ occur as input-output-operators associated with time-delay systems of the form

$$\begin{aligned} \dot{x} &= A(\sigma)x + B(\sigma)u \\ y &= C(\sigma)x + D(\sigma)u \end{aligned} \tag{2.13}$$

where $(A, B, C, D) \in \mathbb{R}[z]^{n\times n + n\times m + p\times n + p\times m}$. Let

$$\mathcal{B}_{\text{ext}}(A, B, C, D) = \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in (\mathcal{C}^\infty)^2 \,\middle|\, \exists\, x \in (\mathcal{C}^\infty)^n : \text{ (2.13) is valid} \right\}$$

be the external $\mathcal{C}^\infty$-behavior of (2.13); see also [Wi, p. 274], where this space is called the manifest or i/o-behavior. For simplicity we use again $\mathcal{C}^\infty$ for the functions only. Using weak solutions it is possible to derive the same result for other function spaces.

In the following characterization we restrict to the strictly proper part of $\mathcal{H}_0$ and include also a description of those $\mathcal{PC}_+^\infty$-functions, which stem from $\mathcal{H}_{0,\mathsf{sp}}$.

Theorem 2.10 *Let $g \in \mathcal{D}'_{\mathsf{c}}$ with $\operatorname{supp} g \subseteq [0, L]$ for some $L \in \mathbb{N}$. Then the following are equivalent:*

i) there exists $q \in \mathcal{H}_{0,\mathsf{sp}}$ such that $g = q(\delta_0^{(1)}, \delta_1)$,

ii) $g \in \mathcal{PC}_+^\infty$ and for every $k \in \{0, \dots, L-1\}$ the restricted function $g|_{(k,k+1]}$ is a finite linear combination of functions from the set $\mathcal{S} := \{t^j e^{\lambda t}(a \sin \mu t + b \cos \mu t) \mid j \in \mathbb{N}_0, \lambda,\ \mu,\ a,\ b \in \mathbb{R}\}$,

iii) there exists a number $n \in \mathbb{N}$ and matrices $(A, b, c) \in \mathbb{R}[z]^{n\times n + n\times 1 + 1\times n}$ satisfying

$$\mathcal{G} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in (\mathcal{C}^\infty)^2 \,\middle|\, y = g * u \right\} = \mathcal{B}_{\text{ext}}(A, b, c, 0).$$

At first glance, part iii) might look suspicious. In general, the external behavior $\mathcal{B}_{\text{ext}}(A, B, C, D)$ consisting of two-sided functions can of course not be described as the graph of an input-output operator. Indeed, if $\ker\left(\frac{d}{dt} I - A(\sigma)\right)$ is not trivial in the underlying function space (which is the case for $\mathcal{C}^\infty$, see [G1, 4.3]), an input u might lead to several outputs y. The only way to still have an input-output operator is via the inclusion $\ker\left(\frac{d}{dt} I - A(\sigma)\right) \subseteq \ker C(\sigma)$, which is in fact the case for the system constructed below. However, the theorem would stay the same if we used one-sided $\mathcal{C}^\infty$-functions only.

PROOF: i) $\Rightarrow$ ii) follows from (2.10) – (2.12) together with Thm. 2.8 i).

ii) $\Rightarrow$ i): By Thm. 2.8 iii) it suffices to show $\mathcal{L}(g) = q^*$ for some $q \in \mathcal{H}_{0,\mathsf{sp}}$. Linearity of $\mathcal{L}$ implies that this follows once the finite Laplace transforms $\hat{\mathcal{L}}(f)(s) := \int_k^{k+1} e^{-st} f(t)dt$ of the functions $f \in \mathcal{S}$ are shown to be in $\{q^* \,|\, q \in \mathcal{H}_{0,\mathsf{sp}}\}$. But this can easily be established upon using the identity $\hat{\mathcal{L}}(t^j g) = (-1)^j \big(\hat{\mathcal{L}}(g)\big)^{(j)}$ and splitting the equation $\hat{\mathcal{L}}(e^{\alpha t}) = (e^{k(\alpha-s)}(e^{\alpha-s} - 1))(\alpha - s)^{-1}$ for $\alpha \in \mathbb{C}$ into its real and imaginary part.

i) $\Rightarrow$ iii): Write $q = \frac{p}{\phi} \in \mathcal{H}_{0,\mathsf{sp}}$ with $\phi(s) = s^n + \sum_{i=0}^{n-1} \phi_i s^i \in \mathbb{R}[s]$ and $p(s,z) = \sum_{i=0}^{n-1} p_i(z) s^i \in \mathbb{R}[s,z]$. Then (2.1) and (2.7) yield $\mathcal{G} = \ker[\widetilde{q}, -1] = \Big\{(u,y)^\mathsf{T} \in (\mathcal{C}^\infty)^2 \,\Big|\, \exists\, x \in \mathcal{C}^\infty : \ \widetilde{\phi} x = u, \ y = \widetilde{p} x\Big\}$. As usual in such situations, putting

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\phi_0 & \cdots & \cdots & -\phi_{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n}, \ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n, \ c = [p_0, \dots, p_{n-1}] \in \mathbb{R}[z]^{1\times n}, \tag{2.14}$$

one gets by straightforward calculation

$$\mathcal{G} = \Big\{ \begin{pmatrix} u \\ y \end{pmatrix} \in (\mathcal{C}^\infty)^2 \,\Big|\, \exists\, w \in (\mathcal{C}^\infty)^n : \ \dot{w} = Aw + bu, \ y = c(\sigma) w \Big\} = \mathcal{B}_{\text{ext}}(A, b, c, 0).$$

iii) $\Rightarrow$ i): By [G2, 3.1a] it is $\mathcal{B}_{\text{ext}}(A,b,c,0) = \ker[\widetilde{p_1}, -\widetilde{p_2}]$ with some $p_i \in \mathcal{H}_0$ and $c(sI - A)^{-1} b = p_2^{-1} p_1 \in \mathbb{R}(s,z)$ being strictly proper in s. Hence the equality

$$\Big\{ \begin{pmatrix} u \\ y \end{pmatrix} \in (\mathcal{C}^\infty)^2 \,\Big|\, p_1(\delta_0^{(1)}, \delta_1) * u = p_2(\delta_0^{(1)}, \delta_1) * y \Big\} = \Big\{ \begin{pmatrix} u \\ y \end{pmatrix} \in (\mathcal{C}^\infty)^2 \,\Big|\, y = g * u \Big\}$$

implies $g = q(\delta_0^{(1)}, \delta_1)$ with $q = p_1 p_2^{-1} \in \mathcal{H}_{0,\mathsf{sp}}$. □

In [G2, Thm. 3.2] it has been shown that the realization procedure given in the proof of i) $\Rightarrow$ iii) can be generalized to matrices $[P, Q] \in \mathcal{H}_0^{p\times(m+p)}$ whenever $Q^{-1}P$ is proper in s and $\det Q \in \mathbb{R}(s,z)$ is monic in s. These conditions can also proven to be essentially necessary for the existence of a first-order latent-variable realization $\ker[\widetilde{P}, \widetilde{Q}] = \mathcal{B}_{\text{ext}}(A, B, C, D)$, see [G2, Prop. 3.2].

Note that the assumption $q \in \mathcal{H}$ instead of $q \in \mathcal{H}_0$ in the above theorem would have led to matrices $(A, b, c) \in \mathbb{R}[z, z^{-1}]^{n^2+n\times 1+1\times n}$, which in turn would have resulted in an advanced system. Actually, with the construction in (2.14) the matrices A and b are constant. Only c involves the shift.

3 Weak coefficient assignability

In this last section we want to discuss the problem of coefficient assignability via (dynamic) state feedback for retarded time-delay systems of the form

$$\dot{x} = A(\sigma)x + B(\sigma)u, \tag{3.1}$$

where A and B are point-delay operators.

Let us start with the following concepts which have been well studied in the literature of systems over rings.

Definition 3.1 *A pair $(A, B) \in \mathbb{R}[z]^{n\times(n+m)}$ or the system (3.1) is said to be*

i) reachable, if the matrix $[sI - A(z), B(z)] \in \mathbb{R}[s, z]^{n\times(n+m)}$ is right invertible over $\mathbb{R}[s, z]$, hence if $\operatorname{rk}[sI - A(z), B(z)] = n$ for all pairs $(s, z) \in \mathbb{C}^2$,

ii) pole assignable, if for all $\lambda_1, \dots , \lambda_n \in \mathbb{R}[z]$ there exists a feedback matrix $F \in \mathbb{R}[z]^{m\times n}$ so that $\det(sI - A - BF) = \prod_{i=1}^{n}(s - \lambda_i)$,

iii) coefficient assignable, if for all monic polynomials $\alpha \in \mathbb{R}[s, z]$ of degree $\deg_s \alpha = n$ there exists a feedback matrix $F \in \mathbb{R}[z]^{m\times n}$ so that $\det(sI - A - BF) = \alpha$. Here and in the sequel monicity refers to the variable s, that is, $\alpha = s^n + \sum_{i=0}^{n-1}\alpha_i(z)s^i$ with polynomials $\alpha_i \in \mathbb{R}[z]$.

It is known from [M, Prop. 1] that a pair (A, B) is reachable if and only if it is pole assignable. On the other hand, coefficient assignability is stronger than reachability. Indeed, one can show by some straightforward calculations that the system $(A, B) = \left(\begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & z^2-1 \end{bmatrix}\right)$ is reachable, but permits no $F \in \mathbb{R}[z]^{2\times2}$ so that $\det(sI - A + BF) = s^2 + s + (z^2 + z + 2)/4 \in \mathbb{R}[z, s]$, [Sch] and [EK1, p. 111].

In the following we will investigate the question of coefficient assignability within a broader class of (dynamic) feedback laws. More precisely, we will allow point delays and distributed delays defined through the operators in $\mathcal{H}_{0,\mathsf{p}}$. By use of this class of control laws it will turn out to be possible to assign arbitrary monic characteristic polynomials of degree n under weaker conditions than the quite restrictive requirement of reachability.

Definition 3.2 *Let $(A, B) \in \mathbb{R}[z]^{n\times(n+m)}$.*

i) [BK] The pair (A, B) is called spectrally controllable, if $\operatorname{rk}[sI - A(e^{-s}), B(e^{-s})] = n$ for all $s \in \mathbb{C}$.

ii) (A, B) is said to be weakly coefficient assignable if for all monic polynomials $\alpha \in \mathbb{R}[s, z]$ with $\deg_s \alpha = n$ there exists a feedback law

$$u = \widetilde{F}x + \widetilde{G}u \tag{3.2}$$

with $F \in \mathcal{H}_{0,\mathsf{p}}^{m\times n}$ and $G \in \mathcal{H}_{0,\mathsf{sp}}^{m\times m}$ such that

$$\det \begin{bmatrix} sI - A & -B \\ -F & I - G \end{bmatrix} = \alpha. \tag{3.3}$$

Remark 3.3

i) From [G1, 3.5] it can be derived that spectral controllability of (A, B) is equivalent to the right invertibility of $[sI - A(z), B(z)]$ as a matrix over $\mathcal{H}_0$. In [OP, Sect. 4] it is even established that in this case $[sI - A(z), B(z)]$ has a right inverse over $\mathcal{H}_{0,\mathsf{sp}}$. The construction of right inverses is basically the same in the two papers, although applied in different situations. One first solves a Bézout-equation over $\mathbb{R}(s)[z]$ and removes in a second step any poles of the corresponding meromorphic functions, see Rem. 2.5. Using a rather different description of $\mathcal{H}_0$, the right-invertibility has also been derived in [KKT2, (3.2)].
Moreover, in [G1, 5.5] and [RW, Thm. 2] it has been shown in different ways that spectral controllability is equivalent to the controllability of the external behavior $\mathcal{B}_{\text{ext}}(A, B, I, 0)$ in the sense of [Wi, V.1]. Furthermore, in [OP] the coincidence with null controllability is established.

ii) By (2.12) and the definition of $\mathcal{H}_{0,\mathsf{p}}$ and $\mathcal{H}_{0,\mathsf{sp}}$ the control law (3.2) is of the form

$$u(t) = \sum_{j=0}^{N} R_j x(t-j) + \int_0^L f(\tau)x(t-\tau)d\tau + \int_0^L g(\tau)u(t-\tau)d\tau \tag{3.4}$$

with $R_j \in \mathbb{R}^{m\times n}$ and where the entries of $f \in (\mathcal{PC}_+^\infty)^{m\times n}$, $g \in (\mathcal{PC}_+^\infty)^{m\times m}$ are of the form as given in Thm. 2.10 ii).

In the literature about time-delay state-space systems treated as infinite-dimensional systems a control problem closely related to weak coefficient assignability has been studied in considerable detail: the question of finite spectrum assignability. This notion refers to the same equation (3.3) but with regard to the following situation. On the one side, only polynomials $\alpha \in \mathbb{R}[s]$ are being considered. This results in a prescribed finite spectrum of the closed loop system, which in most cases is the desirable property. On the other side, a fairly broader class of feedback laws is allowed, namely feedbacks as given in (3.4) but with arbitrary L^2-functions f and g, see e. g. [MO], [WIK, Def. 2.1], [Wa, p. 546], [WNKI, p. 1378], [WNK], and [BL2]. Several results about finite spectrum assignability have been obtained within this context (see again the papers cited above). In particular, in [Wa] it is shown that (3.1) is finite spectrum assignable if and only if it is spectrally controllable. As we will see next, this equivalence still holds true after replacing finite spectrum assignability by weak coefficient assignability. Indeed,

Theorem 3.4 *A pair $(A, B) \in \mathbb{R}[z]^{n\times(n+m)}$ is spectrally controllable if and only if it is weakly coefficient assignable.*

Knowing the results from the literature, this theorem does not come as a surprise. It simply says that all controllers for finite spectrum assignment fall into the class $\mathcal{H}_{0,\mathsf{p}}$ or can be made to do so. Hence, although an infinite-dimensional system, only finitely many parameters need to be found to determine a controller. In Exp. 3.5 it will be shown for a special case how this can be practically accomplished. In the single-input case and for $\alpha \in \mathbb{R}[s]$ the result can also be found in [BL2], the proof being based on the description of $\mathcal{H}_0$ introduced in [KKT2]. We think it is worth giving as short proof within the present framework. It also exhibits that the generalization from finite spectrum to arbitrary closed loop polynomials $\alpha \in \mathbb{R}[s, z]$ is evident in the algebraic setting. The key step in the proof of the multi-input case will be a type of Heymann-Lemma for (3.1), which has been established in [Wa].

Thm. 3.4 also resembles the known characterization of stabilizability of (3.1) via a *stable* right-inverse of $[sI - A(e^{-s}), B(e^{-s})]$, see [EK2, 2.5]. In this case even a finite-dimensional stabilizing compensator exists, see [KKT1, 1.3]

PROOF: Only "$\Rightarrow$" requires proof. Choose a monic $\alpha \in \mathbb{R}[s, z]$ with $\deg_s \alpha = n$.
<u>1. case:</u> $m = 1$
For $j = 1, \ldots, n+1$ denote by $p_j \in \mathbb{R}[s, z]$ the $n \times n$-minor obtained from $[sI - A, -B]$ after deleting the jth column, hence especially $p_{n+1} = \det(sI - A)$. The spectral controllability of (A, B) yields that $p_1, \ldots, p_{n+1}$ are coprime as elements of the Bézout-domain $\mathcal{H}_0$. Thus there exist $r_1, \ldots, r_{n+1} \in \mathcal{H}_0$ such that

$$\alpha = p_{n+1}r_{n+1} - p_n r_n + p_{n-1}r_{n-1} - \ldots + (-1)^n p_1 r_1 = \det \begin{bmatrix} sI - A & -B \\ q & r_{n+1} \end{bmatrix} \tag{3.5}$$

with $q=(r_1,\dots,r_n)\in\mathcal{H}_0^{1\times n}$. According to (2.4), write $q=q_1+d_1$ with $q_1\in\mathcal{H}_{0,\mathsf{sp}}^{1\times n}$ and $d_1\in\mathbb{R}[s,z]^{1\times n}$. Moreover, usual division with remainder applied to the polynomial matrices d_1 and $sI-A$ leads to an equation $d_1=h(sI-A)+d$, where $h\in\mathbb{R}[s,z]^{1\times n}$ and $d\in\mathbb{R}[z]^{1\times n}$. Therefore,

$$\alpha=\det\begin{bmatrix} sI-A & -B \\ q_1+d & r_{n+1}+hB\end{bmatrix}=\det\begin{bmatrix} sI-A & -B \\ \frac{f_1}{\phi} & \frac{c}{\phi}\end{bmatrix} \tag{3.6}$$

where $q_1+d=\frac{f_1}{\phi}\in\mathcal{H}_{0,\mathsf{p}}^{1\times n}$ and $r_{n+1}+hB=\frac{c}{\phi}\in\mathcal{H}_0$. Thus f_1 is a polynomial vector with entries of degree at most $\rho:=\deg\phi$ and c is a scalar polynomial.
Assume $\phi\in\mathbb{R}[s]$ to be monic. Then $\phi\alpha=\det\begin{bmatrix} sI-A & -B \\ f_1 & c\end{bmatrix}$ yields that $c\in\mathbb{R}[s,z]$ is monic and of degree $\deg_s c=\rho$, too. Therefore we can write $\frac{c}{\phi}=1-g$ with some $g\in\mathcal{H}_{0,\mathsf{sp}}$ and the result follows.

<u>2. case:</u> $m>1$
Without restriction suppose the first column b_1 of B to be nonzero. By [Wa, Thm. 2.1] there exists $K\in\mathbb{R}[z]^{m\times n}$ so that $(A+BK,b_1)$ is spectrally controllable. Hence, the 1. case yields $f\in\mathcal{H}_{0,\mathsf{p}}^{1\times n}$ and $g\in\mathcal{H}_{0,\mathsf{sp}}$ satisfying $\alpha=\det\begin{bmatrix} sI-A-BK & -b_1 \\ -f & 1-g\end{bmatrix}$. With

$$F=\begin{bmatrix} f\\0\\ \vdots\\0\end{bmatrix}+\begin{bmatrix} 1-g & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}K\in\mathcal{H}_{0,\mathsf{p}}^{m\times n},\quad G=\operatorname{diag}(g,0,\dots,0)\in\mathcal{H}_{0,\mathsf{sp}}^{m\times m} \tag{3.7}$$

equation (3.3) is obtained. □

In [Wa] it is shown that the matrix K needed for the case $m>1$ can be obtained constructively in a finite number of steps. Together with the procedure given in Rem. 2.5 for the Bézout-equation one can therefore construct the controller algorithmically. A particular simple case arises if the delays occur only in the input, that is, if A is a constant matrix. We exhibit the procedure for this case in the following example.

Example 3.5 We consider the finite spectrum assignment problem, hence $\alpha\in\mathbb{R}[s]$.

a) In the case of a (single-input) spectrally controllable pair $(A,B)\in\mathbb{R}^{n\times n}\times\mathbb{R}[z]^n$ the polynomial $p_{n+1}=\det(sI-A)$ is in $\mathbb{R}[s]$. Therefore, a combination of the procedure of Rem. 2.5 and the above proof yields $\alpha=\det\begin{bmatrix} sI-A & -B \\ f & 1-g\end{bmatrix}$ with a constant $f\in\mathbb{R}^{1\times n}$ and $g\in\mathcal{H}_{0,\mathsf{sp}}$.

We illustrate this with the (unstable) pair $[sI-A,-B]=\left[\begin{array}{cc|c} s & 0 & -z \\ -1 & s-1 & 0\end{array}\right]$ to which we want to assign the stable closed loop polynomial $\alpha=(s+1)(s+2)$. The minors $p_1=z(s-1)$, $p_2=-z$, $p_3=s(s-1)$ are coprime in $\mathcal{H}_0$, so (A,B) is spectrally controllable. Starting with the trivial equation $s(s-1)=0\cdot p_1+0\cdot p_2+1\cdot p_3$, the procedure in Rem. 2.5 requires two steps after which

$$1=-p_1-esp_2+\frac{1+(z-ez)s-z}{s(s-1)}p_3$$

is a Bézout-equation in $\mathcal{H}_0$. Hence

$$\alpha=\det\begin{bmatrix} s & 0 & -z \\ -1 & s-1 & 0 \\ -\alpha & es\alpha & \frac{(1+(z-ez)s-z)\alpha}{s(s-1)}\end{bmatrix}=\det\begin{bmatrix} s & 0 & -z \\ -1 & s-1 & 0 \\ 6e-2 & 6e & 1-\frac{(6ez-2z-4)s+2z-2}{s(s-1)}\end{bmatrix},$$

where the second expression follows after elementary row transformations which produce constants in the first two entries of the last row (this step corresponds to the division with remainder of d_1 by $sI - A$ in the above proof leading to (3.6)). The convolution operator associated with $g = \frac{(6ez-2z-4)s+2z-2}{s(s-1)} = \frac{2(1-z)}{s} + \frac{6(ez-1)}{s-1}$ is obtained from Exp. 2.7 and leads finally to the controller

$$u(t) = (2-6e)x_1(t) - 6ex_2(t) + \int_0^1 (2-6e^\tau)u(t-\tau)d\tau.$$

b) In the case $n = m = 1$ with arbitrary $A = a \in \mathbb{R}$, $B = b(z) \in \mathbb{R}[z]$ and $\alpha = s + \alpha_0 \in \mathbb{R}[s]$ the procedures result in

$$u = -b(e^{-a})^{-1}(a+\alpha_0)x + \widetilde{g}u \quad \text{where} \quad g = (a+\alpha_0)\frac{b(e^{-a})^{-1}b(z) - 1}{s-a} \in \mathcal{H}_{0,\mathsf{sp}}.$$

So e. g. for $b(z) = z^L$ the controller equation simply reads as (see again Exp. 2.7)

$$u = -e^{aL}(a+\alpha_0)x - (a+\alpha_0)\int_0^L e^{a\tau}u(\cdot - \tau)d\tau,$$

which for $L = 1$ has been obtained before with completely different methods in [MO, (2.13),(2.16)].

c) In [MO] even more has been shown. Also in the multi-input case a spectrally controllable pair (A, B) with A being constant and $B = \sum_{j=0}^M B_j z^j$ admits a controller with a constant matrix F. This result cannot simply be derived from our proof. However, the controller given in [MO, Thm. 2.2] is in the class $\mathcal{H}_{0,\mathsf{p}}$; it can be written as in (3.2) with some $F \in \mathbb{R}^{m\times n}$ and $G = F\sum_{j=0}^M (z^jI - e^{-jA})(sI-A)^{-1}B_j$, which is easily seen to be in $\mathcal{H}_{0,\mathsf{sp}}^{m\times m}$ by using the matrix version of (1.4), see [G1, 4.4].

Remark 3.6 It can be shown that the controller given in (3.2) always admits a first-order realization, i. e. one can find matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \mathbb{R}[z]^{r\times r + r\times n + m\times r + m\times n}$ so that $\ker[-\widetilde{F}, I - \widetilde{G}] = \mathcal{B}_{\text{ext}}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, see [G2, Thm. 3.2]. Using such a realization, the equations of the closed loop system are

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{bmatrix} A(\sigma) + (B\hat{D})(\sigma) & (B\hat{C})(\sigma) \\ \hat{B}(\sigma) & \hat{A}(\sigma) \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\ u &= \begin{bmatrix} \hat{D}(\sigma) & \hat{C}(\sigma) \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \end{aligned}$$

This system shows the close connection to the classical framework of dynamical (state-) feedback for state-space systems. It has been studied widely in [H1] with respect to stabilizability.

Under an additional condition on the matrix B one can achieve weak coefficient assignability already with the simpler feedback law $u = \widetilde{F}x$ with $F \in \mathcal{H}_{0,\mathsf{p}}^{m\times n}$.

Corollary 3.7 *Let $(A, B) \in \mathbb{R}[z]^{n\times(n+m)}$ be spectrally controllable and suppose the entries of B are coprime in $\mathbb{R}[z]$. Then, for every monic $\alpha \in \mathbb{R}[s, z]$ with $\deg_s \alpha = n$ there exists $F \in \mathcal{H}_{0,\mathsf{p}}^{m\times n}$ such that*

$$\det \begin{bmatrix} sI - A & -B \\ -F & I \end{bmatrix} = \alpha, \tag{3.8}$$

i. e. the feedback law is given by $u = \widetilde{F}x$.
In particular, the above conditions are met by reachable pairs (A, B).

Proof: Let $U \in Gl_n(\mathbb{R}[z])$ and $V \in Gl_m(\mathbb{R}[z])$ with $\det U = \det V = 1$ and $B_1 := UBV$ is in Smith-form. Then by the condition on B the first row of B_1 is of the form $(\beta, 0, \ldots, 0) \in \mathbb{R}^{1\times m}$ with $\beta \neq 0$. Thus the first column of B_1 is nonzero and as in the proof of Thm. 3.4 we get $\alpha = \det\begin{bmatrix} sI - UAU^{-1} & -B_1 \\ -F & I-G \end{bmatrix}$ with $F \in \mathcal{H}_{0,\mathsf{p}}^{m\times n}$ and $G \in \mathcal{H}_{0,\mathsf{sp}}^{m\times m}$ as in (3.7). The strict properness of $g \in \mathcal{H}_{0,\mathsf{sp}}$ yields $(s - a(z))g \in \mathcal{H}_{0,\mathsf{p}}$ for all $a \in \mathbb{R}[z]$ and hence addition of the first row of $[sI - UAU^{-1}, -B_1]$ multiplied by $\beta^{-1}g$ to the first row of $[-F, I-G]$ leads to

$$\alpha = \det\begin{bmatrix} sI - UAU^{-1} & -UBV \\ -F_1 & I \end{bmatrix} = \det\begin{bmatrix} sI - A & -B \\ -VF_1U & I \end{bmatrix}$$

where the entries of F_1 and consequently those of VF_1U are all in $\mathcal{H}_{0,\mathsf{p}}$. □

We close the paper with the following example.

Example 3.8 Consider again the system $(A, B) = \left(\begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & z^2-1 \end{bmatrix}\right)$ which was seen to be reachable but not coefficient assignable in the sense of Def. 3.1. The pair (A, B) fulfills the conditions of Cor. 3.7. In this case it is easy to obtain for any prescribed polynomial $\alpha = s^2 + as + b \in \mathbb{R}[z][s]$ the controller $F = \begin{bmatrix} a - b\frac{z-1}{s} & b \\ 0 & 0 \end{bmatrix} \in \mathcal{H}_{0,\mathsf{p}}^{2\times 2}$ satisfying (3.8). Hence the feedback law is given by $u_1 = a(\sigma)x_1 + \int_0^1 \big(b(\sigma)x_1\big)(\cdot - \tau)d\tau + b(\sigma)x_2$ and $u_2 = 0$.

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