The Fuhrmann-Realization for Multi-Operator Systems in the Behavioral Context

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Abstract: We use the polynomial approach of Fuhrmann to construct explicit first-order input/latent variable/output realizations for systems with several operators acting on a function space. The class of systems being covered includes certain types of delay-differential, partial differential, and discrete-time mD-systems.

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1 Introduction

In the theory of linear finite-dimensional control systems over a field K the question of realizability for causal transfer functions has been investigated in full detail. It has been known since a long time how to realize a given rational function $G = \sum_{i=0}^{\infty} G_i s^{-i} \in K[\![s^{-1}]\!]^{p\times m}$ as the transfer function $G = C(sI-A)^{-1}B+D$ of a state-space system and what the minimal dimension of such a realization is going to be. As a specific algebraic model for a (minimal) realization the so-called Fuhrmann-realization has been proven very fruitful. It does not realize directly the sequence G_i of coefficients of a transfer function, rather it is based on polynomial factorizations $Q^{-1}P$ of G, see [4] or [5]. This allows a generalization to the behavioral context, i. e. to input/state/output-representations for behaviors given as $\ker[P,Q]$, where $[P,Q]$ denotes a certain operator matrix. For discrete-time behaviors the Fuhrmann-realization has been established in [15, 6]. In [10], Fuhrmann's construction was carried out in a slightly different way for so-called *solution modules*. The latter objects were introduced as a description for certain continuous-time systems over $\mathbb{R}$ and can be regarded as an early version of *behaviors*.

Since the seventies, also linear systems over commutative rings have been studied, see e. g. the surveys [17, 21, 13] and the references therein. The main motivation for these investigations seemed to be the fact, that time-delay systems can be described as differential systems over a (polynomial) ring of delay-operators, see [12]. Ever since, a lot of progress has been made also in the area of realization theory for transfer functions of linear systems over rings [22, 20, 1]. In particular, the Fuhrmann-realization was established for systems over commutative rings, see e. g. [14].

In this paper we want to generalize this construction to linear multi-operator systems in the behavioral framework. The starting point will be a polynomial matrix $[P, Q]$ in several variables representing a list of mutually commuting operators acting on a module $\mathcal{A}$, the latter one serving as the underlying function space for the external variables. This model covers e. g. specific types of delay-differential, partial differential, or discrete-time mD-systems. A list of examples will be given in the next section.

It will be shown that under certain conditions on the matrix $[P, Q]$, the polynomial model of Fuhrmann provides a realization of the behavior $\ker_{\mathcal{A}}[P, Q]$ as an input-output system with a latent variable. This variable is governed by a dynamical equation which is explicit and of first order with respect to one of the several operators. In contrast to onedimensional systems, in the present case the latent variable does not represent (in general) the state of the system, but rather describes the evolution.

The conditions to be imposed on the module $\mathcal{A}$ for the procedure to work are amazingly weak, see equation (2.1). One should bear in mind that after all the classical construction of Fuhrmann is based on i/o-operators acting on Laurent-series, whereas, the behavioral approach to (linear) control theory as introduced by Willems about a decade ago [24] is based essentially on the space of *all* input-output-trajectories of a system (the behavior). With appropriate choices of the function space $\mathcal{A}$, such behaviors comprise in general also the uncontrollable part of the system, hence a subspace which can not be recognized in the transfer function approach. However, the latter approach can be incorporated in the behavioral language as a special case by choosing $\mathcal{A}$ as the space of Laurent-series, see Exp. 2.10.a). The general behavioral description provides therefore a closer look at the situation. Systems with identical transfer functions need not have the same behavior, see Exp. 2.10.b).

Having in mind this gap between the transfer function approach and the behavioral setting, it is the aim of this paper to prove the strength of Fuhrmann's construction. The realization procedure can be generalized straightahead to quite general function modules $\mathcal{A}$ in place of $K(\!(s^{-1})\!)$ and, additionally, to polynomial rings of several operators instead of $K[s]$ only. However, once the procedure is established, it is natural to investigate properties like minimality and uniqueness of the realization. It turns out that with regard to these questions the situation becomes much more involved. In particular, the answers depend on the specific choice of the system class, making a unifying treatment impossible.

We proceed as follows. In the next section the model for the class of multi-operator systems is introduced and justified via a list of examples. Moreover, the concept of behavioral first-order realization is defined and discussed. In Section 3 the construction of Fuhrmann is carried out for this general setting. Finally, in the 4th section some problems about minimality and necessary conditions for realizability are addressed and partial results are given.

2 The polynomial model for the systems description

In this section we shall introduce the polynomial setting and the classes of systems for which the Fuhrmann realization will work. A list of examples covered by this model is given thereafter. These examples will be revisited several times throughout the paper. At the end of this section the notion of behavioral first-order realization is introduced and discussed.

Denote by $K[z_1,\dots,z_l,s]$ the polynomial ring in $l+1$ indeterminates over an arbitrary field K. The indeterminate s is distinguished only because later we will construct realizations which are explicit and of first order with respect to s. For the time being there is no particular meaning to s. We will also use the notation $K[z] := K[z_1,\dots,z_l]$ for the polynomial ring in the first l indeterminates and $K[z,s]$ for $K[z_1,\dots,z_l,s]$.

Let $\mathcal{A}$ be a non-zero divisible $K[z,s]$-module, i. e.

$$p \in K[z,s]\backslash\{0\} \implies \left\{ \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ a & \longmapsto & pa \end{array} \right\} \text{ is surjective.} \tag{2.1}$$

As a consequence, $\mathcal{A}$ is a faithful $K[z,s]$-module, that is,

$$K[z,s] \subseteq \operatorname{End}_{K[z,s]}(\mathcal{A}) \tag{2.2}$$

via the above given multiplication operators.

The situation of a polynomial operator ring acting on a divisible module $\mathcal{A}$ is a special case of the AR-systems studied by Habets, see [9, 4.1]. In [9] the general situation of an operator ring acting on a module is studied with the goal to describe the relationship between different (matrix-) operators having the same kernel.

A matrix $R \in K[z,s]^{p\times q}$ induces the two $K[z,s]$-linear maps

$$\begin{array}{ccc} K[z,s]^q & \longrightarrow & K[z,s]^p \\ v & \longmapsto & Rv \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A}^q & \longrightarrow & \mathcal{A}^p \\ a & \longmapsto & Ra \end{array} \tag{2.3}$$

As we will deal with both maps, we will use the notions $\ker_{K[z,s]} R$ and $\operatorname{im}_{K[z,s]} R$, resp. $\ker_{\mathcal{A}} R$ and $\operatorname{im}_{\mathcal{A}} R$ for the kernel and image of the first resp. second map.

Divisibility of $\mathcal{A}$ generalizes to the matrix case in the usual way.

Lemma 2.1 *Let $R \in K[z,s]^{p\times q}$ be of full row rank and $\mathcal{A}$ be any divisible $K[z,s]$-module. Then $\operatorname{im}_{\mathcal{A}} R = \mathcal{A}^p$.*

The proof is standard, see, e. g., [2, p. 901] or [9, Section 4]. However, to make the paper more self-contained we include the few lines of arguments.

Proof: By the rank assumption we can assume without restriction that R is partitioned as $R = [P,Q]$ with a non-singular matrix $Q \in K[z,s]^{p\times p}$. Let $a = (a_1,\dots,a_p)^\mathsf{T} \in \mathcal{A}^p$ and choose $b = (b_1,\dots,b_p) \in \mathcal{A}^p$ with $(\det Q)b_i = a_i$ for $i = 1,\dots,p$, which is possible by (2.1). Then $a = Q(\operatorname{adj} Q)b = [P,Q]\binom{0}{(\operatorname{adj} Q)b}$, which completes the proof. □

The K-vector space $\mathcal{A}$ should be regarded as a function space with a list of $l+1$ mutually commuting operators acting on it. This model applies in fact to several classes of linear systems, as we will describe next.

Example 2.2 (Delay-Differential Systems)
Let $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and denote by σ_i the shift operator of length $\tau_i > 0$, i. e. $(\sigma_i f)(t) = f(t - \tau_i)$. Then $\mathbb{R}[\sigma_1, \ldots, \sigma_l, \frac{d}{dt}]$ is the ring of all delay-differential operators of the form

$$p = \sum_{\nu=(\nu_1,\ldots,\nu_l)\in\mathbb{N}^l}{}' \sum_{i=0}^{N} p_{\nu,i}\sigma_1^{\nu_1} \circ \ldots \circ \sigma_l^{\nu_l} \circ \frac{d^i}{dt^i},\ p_{\nu,i} \in \mathbb{R} \tag{2.4}$$

where $\sum'$ means this sum being finite. The space $\mathcal{A}$ is an $\mathbb{R}[\sigma_1, \ldots, \sigma_l, \frac{d}{dt}]$-module in the natural way. Precisely, for p as in (2.4) and $f \in \mathcal{A}$ one obtains

$$pf(t) = \sum_{\nu\in\mathbb{N}^l}{}' \sum_{i=0}^{N} p_{\nu,i} f^{(i)}(t - \langle \nu, \tau \rangle),\ t \in \mathbb{R}$$

with $\langle \nu, \tau \rangle = \sum_{j=1}^{l} \nu_j \tau_j$ denoting the usual scalar product. It is obvious that the operators $\sigma_1, \ldots, \sigma_l$, and $\frac{d}{dt} \in \mathrm{End}_{\mathbb{R}}(\mathcal{A})$ mutually commute. Moreover, if $\tau_1, \ldots, \tau_l \in \mathbb{R}$ are chosen to be linearly independent over $\mathbb{Q}$, then $\sigma_1, \ldots, \sigma_l, \frac{d}{dt}$ are algebraically independent elements in the ring $\mathrm{End}_{\mathbb{R}}(\mathcal{A})$, see [12, Sec. 2]. Thus, $\mathbb{R}[\sigma_1, \ldots, \sigma_l, \frac{d}{dt}]$ is a polynomial ring in $l + 1$ indeterminates. Furthermore, it is a deep result in [3, p. 291], that non-zero polynomial delay-differential operators are surjective on $\mathcal{A}$.

The following examples from multidimensional systems theory are studied in detail in the very comprehensive paper [18]. They all constitute so-called large injective cogenerators $\mathcal{A}$ in the category of $K[z, s]$-modules. As this property itself is not needed throughout this paper, we refer the interested reader to [18] for the very definition. More important are the consequences for operators acting on $\mathcal{A}$. We will list some of them in the following example. This will imply in particular property (2.1), which in some of the cases below is, of course, a classical result. In Remark 2.5 we will explain why the delay-differential case is not covered by [18].

Example 2.3 (Multidimensional Systems)
Consider the following situations:

a) $K[z, s] = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{l+1}}]$ and $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^{l+1}, \mathbb{C})$ or $\mathcal{A} = \mathcal{D}'(\mathbb{R}^{l+1})$, the space of complex-valued distributions on $\mathbb{R}^{l+1}$,

b) $K[z, s] = \mathbb{R}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{l+1}}]$ and $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^{l+1}, \mathbb{R})$ or $\mathcal{A} = \mathcal{D}'_{\mathbb{R}}(\mathbb{R}^{l+1})$, the space of real-valued distributions,

c) K any field, $K[z, s] = K[z_1, \ldots, z_{l+1}]$, and $\mathcal{A} := \{\sum_{n\in\mathbb{N}^{l+1}} a(n)t_1^{n_1} \cdot \ldots \cdot t_{l+1}^{n_{l+1}} \mid a(n) \in K\}$ the K-algebra of formal power series in $l + 1$ indeterminates over K. $\mathcal{A}$ becomes a $K[z, s]$-module via the backward shifts with truncation

$$z_i\Big(\sum_{n\in\mathbb{N}^{l+1}} a(n_1, \ldots, n_{l+1})t_1^{n_1} \cdot \ldots \cdot t_{l+1}^{n_{l+1}}\Big) = \sum_{n\in\mathbb{N}^{l+1}} a(n_1, \ldots, n_i + 1, \ldots, n_{l+1})t_1^{n_1} \cdot \ldots \cdot t_{l+1}^{n_{l+1}}.$$

This is usually the framework being used in the study of discrete-time mD-systems [25]. One may allow K to be a finite field as it is done in coding theory [23].

It is the main result of [18] that all situations above have some strong algebraic structure in common, see [18, (54) p. 33]. As a consequence, operators acting on $\mathcal{A}$ show some nice features, resembling the situation of ordinary differential or difference equations. We only list the following:

(1) [18, (46), p. 30] For $P \in K[z,s]^{n\times m}$ and $Q \in K[z,s]^{l\times n}$ one has $\ker_{K[z,s]} P^\mathsf{T} = \operatorname{im}_{K[z,s]} Q^\mathsf{T} \Longleftrightarrow \ker_{\mathcal{A}} Q = \operatorname{im}_{\mathcal{A}} P$.

(2) In particular, if $P \in K[z,s]^{n\times m}$ has rank n, then $\operatorname{im}_{\mathcal{A}} P = \mathcal{A}^n$.

(3) [18, (61), p. 36] For $P \in K[z,s]^{n\times m}$ and $R \in K[z,s]^{r\times m}$ it is $\ker_{\mathcal{A}} P \subseteq \ker_{\mathcal{A}} R \Longleftrightarrow R = XP$ for some $X \in K[z,s]^{r\times n}$.

For the main part of this paper, only (2) is needed as it implies the divisibility of $\mathcal{A}$. In Remark 2.5 we will see that (1) is not satisfied for delay-differential systems, explaining why this case is not covered by [18]. We will use (1)–(3) later in the examples when discussing multidimensional systems in more detail, see 2.10.c) and 4.3.

Example 2.4 (Transfer Functions)
Trivial examples for non-zero divisible $K[z,s]$-modules are, of course, $\mathcal{A} = K(z,s)$ and $\mathcal{A} = K(z)((s^{-1})) = \{\sum_{i=-\infty}^{N} f_i s^i \mid N \in \mathbb{Z},\ f_i \in K(z)\}$ with the natural $K[z,s]$-module structure. In this case, behavioral theory coincides with the transfer function framework as we will see in Exp. 2.10.a).

Remark 2.5 We briefly illustrate why the situation discussed in [18] does not cover delay-differential systems, cf. [18, p. 17]. It suffices to consider systems with commensurate delays only. Hence, let $K[z,s] = \mathbb{R}[z_1,s]$ with $z_1 f(t) = f(t-1)$ and $sf = \dot{f}$. Let $\mathcal{A}$ be the $\mathbb{R}[z_1,s]$-module $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Consider the following matrices

$$P = \begin{bmatrix} z_1 - 1 \\ s \end{bmatrix} \in \mathbb{R}[z_1,s]^{2\times 1},\ Q = [s, 1-z_1] \in \mathbb{R}[z_1,s]^{1\times 2}.$$

Note that $\ker_{\mathbb{R}[z_1,s]} P^\mathsf{T} = \operatorname{im}_{\mathbb{R}[z_1,s]} Q^\mathsf{T}$, while $\operatorname{im}_{\mathcal{A}} P \subsetneqq \ker_{\mathcal{A}} Q$ as can be seen by the constant function $w = (0,1)^\mathsf{T} \in \mathcal{A}^2$. Therefore, property (1) in Exp. 2.3 is violated showing that $\mathcal{A}$ is not a large injective cogenerator (actually, this means that $\mathcal{A}$ is not an injective $\mathbb{R}[z_1,s]$-module).

Remark 2.6 So far, it does not play any role having one of the variables distinguished. Even more, if $x_1,\dots,x_{l+1}$ are algebraically independent elements over K, the same is true for $y_1,\dots,y_{l+1}$, where

$$(y_1,\dots,y_{l+1})^\mathsf{T} = A(x_1,\dots,x_{l+1})^\mathsf{T} + (b_1,\dots,b_{l+1})^\mathsf{T}$$

for some $A \in Gl_{l+1}(K)$ and $b \in K^{l+1}$. In particular, $K[y_1,\dots,y_{l+1}] = K[x_1,\dots,x_{l+1}]$.
E. g. in Exp. 2.2, the polynomial ring can also be described as $\mathbb{R}[\frac{d}{dt}, \sigma_1 - 1, \dots, \sigma_l - 1]$, where we replaced the shift operators by the corresponding difference operators and changed the ordering of the indeterminates. In this case, the list of operators $(z_1,\dots,z_l,s)$ reads as $(\frac{d}{dt}, \sigma_1 - 1, \dots, \sigma_l - 1)$, so that $s = \sigma_l - 1$ is the distinguished operator. The procedure of the next section would hence result in a realization with respect to the last difference operator $\sigma_l - 1$ in place of the differential operator $\frac{d}{dt}$ as being indicated in the original situation of Exp. 2.2.

Let us return to the general case with a divisible $K[z,s]$-module $\mathcal{A}$. For $R \in K[z,s]^{p\times q}$ the kernel $\ker_{\mathcal{A}} R$ is a submodule of $\mathcal{A}^q$ and can be viewed as the behavior of a dynamical system in the sense of [24], only one has to accept the multidimensional time-axis $\mathbb{R}^{l+1}$ or $\mathbb{N}^{l+1}$ in Exp. 2.3. In this sense, if R is given as $R = (R_{ij})_{\substack{i=1,\dots,p \\ j=1,\dots,q}}$, the associated behavior

$$\ker_{\mathcal{A}} R = \left\{ (a_1,\dots,a_q)^\mathsf{T} \in \mathcal{A}^q \,\middle|\, \sum_{j=1}^{q} R_{ij} a_j = 0 \text{ for } i = 1,\dots,p \right\}$$

is made of all trajectories in $\mathcal{A}^q$ which are governed by a system of (higher order) equations, e. g., delay-differential equations, partial differential equations, or partial difference equations. The q coordinate functions of $(a_1, \dots, a_q)^\mathsf{T} \in \ker_\mathcal{A} R$ are called the *external variables* of the behavior.

It is the aim of this paper to show that under specific circumstances the behavior $\ker_\mathcal{A} R$ can be represented as the *external behavior* of an explicit first-order input/latent variable/output system. More precisely, the following version of realizability will be investigated.

Definition 2.7 *A matrix $R \in K[z,s]^{p\times(m+p)}$ is called realizable, if there exists a number $n \in \mathbb{N}$ and matrices $(A,B,C,D) \in K[z]^{n^2+nm+pn+pm}$ so that*

$$\ker_\mathcal{A} R = \mathcal{B}_{\text{ext}}(A,B,C,D) := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{A}^{m+p} \,\middle|\, \exists\, x \in \mathcal{A}^n : \begin{array}{rcl} sx & = & Ax + Bu \\ y & = & Cx + Du \end{array} \right\}. \tag{2.5}$$

In case such matrices exist, we call the quadruple (A,B,C,D) a realization of R.

Remark 2.8 Equation (2.5) indicates, that the first m of the (external) variables in $\ker_\mathcal{A} R$ have the usual properties of inputs, while the last p variables can be regarded as outputs of the system. In fact, by the surjectivity of full row rank matrices, for every $u \in \mathcal{A}^m$ there exists some (non-unique) trajectory $y \in \mathcal{A}^p$ so that $(u^\mathsf{T}, y^\mathsf{T})^\mathsf{T} \in \ker_\mathcal{A} R$. A reordering of the columns of R would reflect the possibility of a different partition of the $m+p$ external variables into inputs and outputs. This additional freedom of choice would be more adequate in the behavioral approach. We will assume that such a reordering has already been carried out.
Anyway, realizability in the sense of Def. 2.7 requires the system to have exactly p outputs, where p is the number of equations governing the system. The remaining external variables form a maximal set of inputs. As we will see in Exp. 2.10 and in Exp. 4.3, this notion of realizability implies in most cases R to be of rank p, and thus of full row rank if of the size $p \times (m+p)$. Hence, while R might be realizable, $\begin{bmatrix} R \\ 0 \end{bmatrix}$ certainly is not, although $\ker_\mathcal{A} R = \ker_\mathcal{A} \begin{bmatrix} R \\ 0 \end{bmatrix}$. Actually, we will be imposing the full row rank condition on R for the realization procedure. Therefore, our considerations are restricted to behaviors which admit a full row rank kernel representation (which we have at our disposal for the realization procedure) and the above little example $\ker_\mathcal{A} \begin{bmatrix} R \\ 0 \end{bmatrix}$ is not really excluded. However, this restriction is indeed crucial: since $K[z,s]$ is not a principal ideal domain, it is in general not possible to eliminate linearly dependent rows of R without changing the associated behavior $\ker_\mathcal{A} R$. We will study the above examples 2.2 - 2.4 at the end of this section under this point of view.

As it is well known, realization of transfer functions and of behaviors are in general not the same thing, see also Exp. 2.10.b) However, the following relationship can be proven. The second of the following statements will be crucial later as it relates polynomial equations with operator identities on $\mathcal{A}$. It is a generalization of the purely differential version given in [19, Lemma 2.1].

Proposition 2.9 *Let $(A,B,C,D) \in K[z]^{n^2+nm+pn+pm}$ and $R = [P,Q] \in K[z,s]^{p\times(m+p)}$ be of rank p.*
a) If (2.5) is valid, then Q is non-singular and

$$-Q^{-1}P = C(sI-A)^{-1}B + D. \tag{2.6}$$

In particular, $Q^{-1}P$ is a matrix over the ring $K[z][\![s^{-1}]\!]^{p\times m}$ of power series in s^{-1} with coefficients from $K[z]$.

b) Suppose that $X := -QC(sI-A)^{-1} \in K[z,s]^{p\times n}$, i.e. is polynomial, and that the polynomial matrix $[X,R] \in K[z,s]^{p\times(n+m+p)}$ is right-invertible over $K[z,s]$. Then equation (2.6) implies

$$\ker_{\mathcal{A}}[P,Q] = \mathcal{B}_{\text{ext}}(A,B,C,D)$$

for any non-zero divisible $K[z,s]$-module $\mathcal{A}$.

Recall that right-invertibility of $[X,R]$ as a matrix over $K[z,s]$ is identical to zero primeness of $[X,R]$, i. e., $\text{rk}\,[X(\lambda_1,\ldots,\lambda_{l+1}), R(\lambda_1,\ldots,\lambda_{l+1})] = p$ for all $(\lambda_1,\ldots,\lambda_{l+1}) \in \bar{K}^{l+1}$, where $\bar{K}$ denotes the algebraic closure of the field K; this is an easy consequence of Hilbert's Nullstellensatz, see e. g. [27, p. 161] for the matrix version.

Proof: a) From (2.5) one can derive the equation

$$M := QC\text{adj}\,(sI-A)B + \det(sI-A)QD + \det(sI-A)P = 0. \tag{2.7}$$

In fact, by (2.2) it is enough to show that $Mu = 0$ for all $u \in \mathcal{A}^m$. Thus, let $u \in \mathcal{A}^m$ be arbitrary and choose $x \in \mathcal{A}^n$ so that $Bu = (sI-A)x$, see Lemma 2.1. Put $y = Cx + Du$. Then $Pu + Qy = 0$ and

$$\begin{aligned} &QC\text{adj}\,(sI-A)Bu + \det(sI-A)QDu + \det(sI-A)Pu \\ &= QC\text{adj}\,(sI-A)(sI-A)x + \det(sI-A)QDu + \det(sI-A)Pu \\ &= \det(sI-A)(QCx + QDu + Pu) \\ &= 0, \end{aligned}$$

which proves $Mu = 0$. Equation (2.7) implies

$$[P,Q]\begin{bmatrix} I_m \\ C(sI-A)^{-1}B + D \end{bmatrix} = 0,$$

considered as an equation over the field $K(z,s)$. Since both matrices are of full rank, this implies $\det Q \neq 0$ as well as (2.6).

b) By Serre's conjecture on projective modules over polynomial rings proven by Quillen/Suslin, see [16, pp. 490] or [26, p. 513], the matrix $[X,R]$ can be completed to a unimodular matrix

$$\begin{bmatrix} U_1 & U_2 \\ X & R \end{bmatrix} \in Gl_{n+m+p}(K[z,s]).$$

Using $R = [P,Q]$, the assumptions can be restated as the matrix identity

$$\begin{bmatrix} U_1 & U_2 \\ X & R \end{bmatrix}\begin{bmatrix} sI-A & -B \\ 0 & I_m \\ C & D \end{bmatrix} = \begin{bmatrix} T \\ 0 \end{bmatrix},$$

where $T = U_1[sI-A,-B] + U_2\begin{bmatrix} 0 & I_m \\ C & D \end{bmatrix} \in K[z,s]^{(n+m)\times(n+m)}$. Since $\det T \neq 0$, the associated operator is surjective on $\mathcal{A}^{n+m}$ by Lemma 2.1 and we may argue as follows:

$$\begin{aligned} \mathcal{B}_{\text{ext}}(A,B,C,D) &= \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{A}^{m+p} \,\middle|\, \exists\, \xi \in \mathcal{A}^{n+m} : \begin{pmatrix} 0 \\ u \\ y \end{pmatrix} = \begin{bmatrix} sI-A & -B \\ 0 & I_m \\ C & D \end{bmatrix} \xi \right\} \\ &= \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{A}^{m+p} \,\middle|\, \exists\, \xi \in \mathcal{A}^{n+m} : \begin{bmatrix} U_2 \\ R \end{bmatrix}\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} T \\ 0 \end{pmatrix} \xi \right\} \\ &= \ker_{\mathcal{A}} R = \ker_{\mathcal{A}}[P,Q]. \end{aligned}$$

□

We close this section with a discussion of the full row rank condition of Prop. 2.9 in the main examples 2.2-2.4.

Example 2.10

a) (Transfer Functions) Let K be any field and consider $\mathcal{A} = K(z, s)$ or $\mathcal{A} = K(z)((s^{-1}))$, both of which are $K[z, s]$-modules in the natural way, see Exp. 2.4. Then

$$\begin{aligned}\mathcal{B}_{\text{ext}}(A, B, C, D) &= \left\{ \binom{u}{y} \in \mathcal{A}^{m+p} \,\middle|\, y = (C(sI - A)^{-1}B + D)u \right\} \\ &= \ker_{\mathcal{A}}[-C(sI - A)^{-1}B + D, I_p] = \ker_{\mathcal{A}}[P, Q]\end{aligned}$$

where $-Q^{-1}P = C(sI - A)^{-1}B + D$ is any factorization of the transfer function into polynomial matrices (which, of course, exists). Thus, in this case, $\mathcal{B}_{\text{ext}}(A, B, C, D)$ admits a full row rank kernel representation $[P, Q] \in K[z, s]^{p\times(m+p)}$. Obviously, for this special choice of $\mathcal{A}$ realization of behaviors is identical to realization of transfer functions.

b) (Delay-Differential Systems) In the case of Exp. 2.2 above, where $s = \frac{d}{dt}$ and $z_1, \dots, z_l$ are shift operators of noncommensurate lengths $\tau_1, \dots, \tau_l$, a realization as in (2.5) results in a time-delay system of the form

$$\begin{aligned}\dot{x}(t) &= {\sum_{\nu\in\mathbb{N}^l}}' A_\nu x(t - \langle \nu, \tau\rangle) + {\sum_{\nu\in\mathbb{N}^l}}' B_\nu u(t - \langle \nu, \tau\rangle) \\ y(t) &= {\sum_{\nu\in\mathbb{N}^l}}' C_\nu x(t - \langle \nu, \tau\rangle) + {\sum_{\nu\in\mathbb{N}^l}}' D_\nu u(t - \langle \nu, \tau\rangle)\end{aligned}$$

with constant matrices A_ν, B_ν, C_ν, and D_ν.

The special case $l = 1$, i. e., commensurate delays, has been investigated in detail in [7, 8]. From these papers one can deduce the following results concerning realizability.

i) [8, Thm. 3.1 (a)] The behavior $\mathcal{B}_{\text{ext}}(A, B, C, D)$ with sizes as in Def. 2.7 does always admit a full row rank kernel representation, hence $\mathcal{B}_{\text{ext}}(A, B, C, D) = \ker_{\mathcal{A}}[\hat{P}, \hat{Q}]$ for some full row rank matrix $[\hat{P}, \hat{Q}] \in \mathbb{R}[z_1, s]^{p\times(m+p)}$.

ii) [7, Prop. 4.4] If $R = [P, Q] \in \mathbb{R}[z_1, s]^{p\times(m+p)}$ satisfies (2.5), then i) yields $[P, Q] = W[\hat{P}, \hat{Q}]$ with some non-singular $W \in \mathbb{R}(s)[z_1, z_1^{-1}]^{p\times p}$, thus $\operatorname{rk}[P, Q] = p$. (Actually, W is unimodular over some subring $\mathcal{H}$ of $\mathbb{R}(s)[z_1, z_1^{-1}]$ as investigated in [7].) Moreover, $\det Q$ is monic after some always possible normalization of $[P, Q]$, see [8, Prop. 3.2]. The normalization consists of premultiplication of $[P, Q]$ with some $V \in Gl_p(\mathbb{R}[s, z_1, z_1^{-1}])$. Due to bijectivity of the shift operator z_1 on $\mathcal{A}$, this does not change $\ker_{\mathcal{A}}[P, Q]$.

Hence, in this special case too, the full row rank assumption on $[P, Q]$ is no restriction. It remains an open question whether this is true for the case of incommensurate delays as well, see [28, p. 234].

Although it is well known that in general realization of behaviors is not identical to realization of transfer functions, we want to illustrate this fact by the following trivial example with commensurate delays.

$$\begin{aligned}\ker_{\mathcal{A}}[z_1 - 1, -\tfrac{d}{dt}] &= \left\{ \binom{u}{y} \in \mathcal{A}^2 \,\middle|\, \exists\, x \in \mathcal{A} : \dot{x} = (z_1 - 1)u,\ y = x \right\} \\ &\supsetneq \left\{ \binom{u}{y} \in \mathcal{A}^2 \,\middle|\, \exists\, x \in \mathcal{A} : \dot{x} = u,\ y = (z_1 - 1)x \right\}\end{aligned}$$

(note that $(u, y)^\mathsf{T} = (0, 1)^\mathsf{T}$ is contained in the first but not in the second external behavior). Both first order systems have transfer function $\frac{z_1-1}{s}$.

c) (Multidimensional Systems) In the multidimensional case of Exp. 2.3, the restriction to behaviors with a full row rank kernel representation is indeed crucial as can be seen from the following example. Let $\frac{\partial}{\partial x_i} = \partial_i$ and $K[z,s] = \mathbb{C}[\partial_1, \partial_2, \partial_3]$ act on $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$ as in 2.3, especially $s = \partial_3$ is the distinguished variable. It is easy to see that

$$\ker_{\mathbb{C}[\partial_1,\partial_2,\partial_3]} \begin{bmatrix} 0 & \partial_2 & -\partial_1 & \partial_3 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \operatorname{im}_{\mathbb{C}[\partial_1,\partial_2,\partial_3]} \begin{bmatrix} \partial_2 & -\partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \\ 0 & \partial_3 & \partial_2 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

Thus, property (1) of 2.3 implies

$$\operatorname{im}_{\mathcal{A}} \left[\begin{array}{cc} 0 & 1 \\ \partial_2 & 0 \\ -\partial_1 & 0 \\ \hline \partial_3 & 1 \end{array}\right] = \ker_{\mathcal{A}} \left[\begin{array}{ccc|c} \partial_2 & \partial_3 & 0 & -\partial_2 \\ -\partial_1 & 0 & \partial_3 & \partial_1 \\ 0 & \partial_1 & \partial_2 & 0 \end{array}\right]$$

and from this one obtains the realization

$$\ker_{\mathcal{A}} M := \ker_{\mathcal{A}} \begin{bmatrix} \partial_2 & \partial_3 & 0 \\ -\partial_1 & 0 & \partial_3 \\ 0 & \partial_1 & \partial_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \partial_2 & 0 \\ -\partial_1 & 0 \end{bmatrix} \ker_{\mathcal{A}}[\partial_3, 1] = \mathcal{B}_{\text{ext}}\Big(0, -1, \begin{bmatrix} \partial_2 \\ -\partial_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big)$$

If $\ker_{\mathcal{A}} M$ had a full row rank kernel representation, say $\ker_{\mathcal{A}} M = \ker_{\mathcal{A}} \hat{M}$ with $\hat{M} \in \mathbb{C}[\partial_1, \partial_2, \partial_3]^{2\times 3}$, property (3) of 2.3 would imply that $\operatorname{im}_{\mathbb{C}[\partial_1,\partial_2,\partial_3]} M^\mathsf{T} = \operatorname{im}_{\mathbb{C}[\partial_1,\partial_2,\partial_3]} \hat{M}^\mathsf{T}$ is a free module. But this is certainly not the case. Hence there exist behaviors which do admit realizations in the sense of Def. 2.7, but which do not have a full row rank kernel representation. Behaviors of this type are excluded from our construction.
However, it will be seen in Exp. 4.3 that each behavior $\mathcal{B}_{\text{ext}}(A, B, C, D)$ with the sizes as in Def. 2.7 does have kernel representation of rank p. As a consequence, (2.5) implies $\operatorname{rk} R = p$ for multidimensional systems, too.

3 The realization procedure of Fuhrmann

We will now establish Fuhrmann's realization in the setting introduced in the previous section. Thus, throughout the construction, $K[z,s] = K[z_1, \ldots, z_l, s]$ is a polynomial ring in $l+1$ indeterminates acting on a nonzero divisible $K[z,s]$-module $\mathcal{A}$. The algebraic independence of the operators $z_1, \ldots, z_l$, and s will be crucial as it enables us to apply the Theorem of Quillen/Suslin. The divisibility will be needed to eliminate the latent variable of a first-order realization.

Let us first fix some notations.

Definition 3.1

a) *With* $K[z]((s^{-1})) := \{\sum_{i=-\infty}^{N} f_i s^i \mid N \in \mathbb{Z},\ f_i \in K[z]\}$ *we denote the ring of formal Laurent-series in* s^{-1} *with coefficients in the polynomial ring* $K[z]$*. This ring contains the ring* $K[z][[s^{-1}]]$ *of formal power series in* s^{-1} *as well as the polynomial ring* $K[z,s]$*.*

b) *For a matrix* $F = \sum_{i=-\infty}^{N} F_i s^i \in K[z]((s^{-1}))^{p\times k}$ *with* $F_i \in K[z]^{p\times k}$ *and* $F_N \neq 0$ *we write* $\deg F := N$ *for the degree of* F *with respect to* s*.*

c) An element $F \in K[z]((s^{-1}))^{p\times k}$ *is called proper (resp. strictly proper) if* $\deg F \le 0$ *(resp.* $\deg F < 0$*). Hence* F *is (strictly) proper iff all its entries are (strictly) proper.*

d) Define the maps Π_- *and* Π_+ *as the projections onto the strictly proper part and polynomial part respectively, that is*

$$\begin{aligned}\Pi_- :\ K[z]((s^{-1}))^{p\times k} &\longrightarrow K[z]((s^{-1}))^{p\times k} \\ \sum_{i=-\infty}^{N} F_i s^i &\longmapsto \sum_{i=-\infty}^{-1} F_i s^i\end{aligned} \qquad \begin{aligned}\Pi_+ :\ K[z]((s^{-1}))^{p\times k} &\longrightarrow K[z]((s^{-1}))^{p\times k} \\ \sum_{i=-\infty}^{N} F_i s^i &\longmapsto \sum_{i=0}^{N} F_i s^i\end{aligned}$$

Note that $\Pi_+ = \mathrm{id} - \Pi_-$*.*

e) A non-zero element $f = \sum_{i=-\infty}^{N} f_i s^i \in K[z]((s^{-1}))$ *is called monic, if its highest coefficient is a non-zero constant, i. e. if* $f_N \in K\backslash\{0\}$*.*

The starting point for the construction will be a matrix

$$[P, Q] \in K[z, s]^{p\times(m+p)} \text{ with } \det Q \text{ being monic and } Q^{-1}P \text{ being strictly proper.} \tag{3.1}$$

As we saw in Exp. 2.10, the restriction to full row rank matrices $[P, Q]$ is not crucial in the case of transfer functions and delay-differential systems with commensurate delays. The condition $\det Q$ being monic is necessary for the specific construction to work. Whether this condition is necessary in general for realizability of $\ker_{\mathcal{A}}[P, Q]$ will be discussed in the last section. The restriction to strictly proper transfer matrices instead of only proper ones will hold notations simple. The proper case can be derived easily by use of the equivalence

$$\ker_{\mathcal{A}}[P, Q] = \mathcal{B}_{\text{ext}}(A, B, C, D) \Longleftrightarrow \ker_{\mathcal{A}}[P + QD, Q] = \mathcal{B}_{\text{ext}}(A, B, C, 0).$$

On the conditions (3.1) we will show that the classical construction of Fuhrmann constitutes indeed a realization in the behavioral sense of Def. 2.7. This is not a priori clear, since the Fuhrmann-realization gives a polynomial model $(A, B, C, D) \in K[z]^{n^2+nm+pn+pm}$ for the transfer function identity $-Q^{-1}P = C(sI - A)^{-1}B + D$. As we saw in Exp. 2.10.a), the transfer function identity can be viewed as a behavioral one in the special case $\mathcal{A} = K(z, s)$. It is quite amazing that there is only little extra work to do for showing that the construction leads also to a behavioral realization for arbitrary nonzero divisible modules $\mathcal{A}$.

The first step of the Fuhrmann-realization of $[P, Q]$ is the construction of an abstract "state-module", this terminus to be understood in the language of systems over rings. This space is going to be a $K[z]$-module depending only on the matrix Q. Define the map

$$\begin{aligned}\Pi_Q : K[z, s]^p &\longrightarrow K[z, s]^p \\ f \quad &\longmapsto Q\Pi_-(Q^{-1}f)\end{aligned}$$

Note that the monicity of $\det Q$ guarantees that $Q^{-1} \in K[z]((s^{-1}))^{p\times p}$. Especially, $\Pi_Q(f) = f - Q\Pi_+(Q^{-1}f)$ is indeed in $K[z, s]^p$. Obviously, Π_Q is a $K[z]$-linear map satisfying $\Pi_Q \circ \Pi_Q = \Pi_Q$, whence a projection.

Theorem 3.2 *Let* $[P, Q]$ *be as in (3.1). Define the* $K[z]$*-module* $S_Q := \operatorname{im} \Pi_Q \subseteq K[z, s]^p$*. Then* S_Q *satisfies the following properties.*

a) $S_Q = \{f \in K[z, s]^p \mid Q^{-1}f \text{ is strictly proper}\}$, *b)* $S_Q = \operatorname{span}_{K[z]}\{\Pi_Q(e_i s^j) \mid i = 1, \dots, p,\ j = 0, \dots, \deg\det Q - 1\}$, *where* $e_1, \dots, e_p$ *are the standard basis vectors of* $K[z]^p$,

c) $K[z,s]^p = \ker \Pi_Q \oplus S_Q = QK[z,s]^p \oplus S_Q$,

d) S_Q *is a free* $K[z]$*-module with* $\operatorname{rank} S_Q = \deg\det Q$.

The parts a) – c) can also be found in [14]. Part d) is the main step for the procedure, since a free state module allows matrix representations for linear operators.

Proof: a) is obvious.
b) Let $f \in K[z,s]^p$. We can conduct long division by $\det Q$ and get an expression $f = f_1 \det Q + g_1$ with $f_1,\ g_1 \in K[z,s]^p$ and $\deg g_1 < \deg\det Q$. Then

$$\Pi_Q(f) = Q\Pi_-\big(\operatorname{adj}(Q) f_1 + Q^{-1} g_1\big) = Q\Pi_-(Q^{-1} g_1) = \Pi_Q(g_1),$$

hence $S_Q = \{\Pi_Q(g) \mid g \in K[z,s]^p,\ \deg g < \deg\det Q\}$, which yields b).
c) The first equality holding true in general for projections, it remains to prove $\ker \Pi_Q = QK[z,s]^p$. The inclusion "$\supseteq$" follows directly from the definition of Π_Q. For "$\subseteq$" let $f \in K[z,s]^p$ with $\Pi_Q(f) = 0$. Then $0 = Q(\mathrm{id} - \Pi_+)(Q^{-1}f) = f - Q\Pi_+(Q^{-1}f)$, which shows that $f = Q\Pi_+(Q^{-1}f)$ is contained in $QK[z,s]^p$.
d) By b) and c) the $K[z]$-module S_Q is finitely generated and projective. Hence the Theorem of Quillen/Suslin states that S_Q is also a free $K[z]$-module, see [16, p. 492].
Let $\{g_1, \dots, g_n\} \subseteq K[z,s]^p$ be a basis of S_Q. In order to show that $n = \deg\det Q$, we will use the results about the Fuhrmann-realization over fields, in this case over the field $K(z)$. Thus consider the projection

$$\begin{aligned} \hat{\Pi}_Q : K(z)[s]^p &\longrightarrow K(z)[s]^p \\ f \quad &\longmapsto Q\Pi_-(Q^{-1}f) \end{aligned}$$

where, of course, Π_- is used for the projection onto the strictly proper part in $K(z)(\!(s^{-1})\!)^p$ as well. Put

$$\hat{S}_Q := \operatorname{im} \hat{\Pi}_Q \subseteq K(z)[s]^p. \tag{3.2}$$

Denote by $\bar{Q} = \operatorname{diag}(q_1, \dots, q_p) \in K(z)[s]^{p\times p}$ the Smith-form of Q over $K(z)[s]$. As in part a) of this theorem it is

$$\begin{aligned} \hat{S}_{\bar{Q}} &= \{f \in K(z)[s]^p \mid \bar{Q}^{-1} f \text{ strictly proper}\} \\ &= \{(f_1, \dots, f_p)^\mathsf{T} \in K(z)[s]^p \mid \deg f_i < \deg q_i,\ i = 1, \dots, p\}. \end{aligned}$$

Since [5, Thm. 4.11] or direct calculations tell us that $\hat{S}_{\bar{Q}}$ and $\hat{S}_Q$ are isomorphic $K(z)$-vector spaces, we obtain $\dim_{K(z)} \hat{S}_Q = \sum_{i=1}^p \deg q_i = \deg\det Q$. On the other side, the $K(z)$-linearity of $\hat{\Pi}_Q$ implies $\hat{S}_Q = \operatorname{span}_{K(z)}\{g_1, \dots, g_n\}$. Indeed, if $g = \hat{\Pi}_Q(f)$ with $f = h^{-1}\bar{f}$, $\bar{f} \in K[z,s]^p$ and $h \in K[z]\backslash\{0\}$, then $g = h^{-1}\hat{\Pi}_Q(\bar{f}) = h^{-1}\Pi_Q(\bar{f}) = h^{-1}\bar{g}$ with some $\bar{g} \in S_Q$.
Together with the linear independence of $g_1, \dots, g_n$ over $K(z)$, this amounts to

$$\operatorname{rank} S_Q = n = \dim_{K(z)} \hat{S}_Q = \deg\det Q. \qquad \square$$

Now we are able to establish the Fuhrmann-realization.

Theorem 3.3 *Let $[P,Q]$ be as in (3.1) with* $\deg\det Q = n$.
Define the $K[z]$-linear maps

$$\hat{A}:\ S_Q \longrightarrow S_Q \qquad \hat{B}:\ K[z]^m \longrightarrow S_Q \qquad \hat{C}:\ S_Q \longrightarrow K[z]^p$$
$$f \longmapsto \Pi_Q(sf) \qquad \xi \longmapsto -P\xi \qquad f \longmapsto \Pi_+(Q^{-1}sf)$$

Fix a basis $f_1, \ldots, f_n \in K[z,s]^p$ of S_Q and let $A \in K[z]^{n\times n}$, $B \in K[z]^{n\times m}$, and $C \in K[z]^{p\times n}$ be the matrix representations of $\hat{A}$, $\hat{B}$, and $\hat{C}$ with respect to the chosen basis of S_Q and the standard bases of $K[z]^m$ and $K[z]^p$. Then

$$\mathcal{B}_{\text{ext}}(A,B,C,0) = \ker_{\mathcal{A}}[P,Q]$$

for any non-zero divisible $K[z,s]$-module $\mathcal{A}$.
Furthermore, the realization is coreachable in the sense of systems over the ring $K[z]$, that is

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} \text{ is left-invertible over } K[z,s].$$

Proof: First of all, one should note that by the strict properness of $Q^{-1}P$ the image of the map $\hat{B}$ is in fact contained in S_Q, see Thm. 3.2.a). Furthermore, by definition of S_Q the vector $Q^{-1}sf$ is proper. Hence, $\hat{C}(f)$ is simply the constant part of $Q^{-1}sf$ and indeed in $K[z]^p$.
From the proof of Thm. 3.2.d) we know that $f_1, \ldots, f_n$ is a basis of the $K(z)$-vector space $\hat{S}_Q$ in (3.2) as well. Hence the triple (A,B,C) regarded as matrices over $K(z)$ constitutes also the Fuhrmann-realization of $[P,Q] \in K(z)[s]^{p\times(m+p)}$ and [5, Thm. 10.1] tells us that

$$C(sI-A)^{-1}B = -Q^{-1}P. \tag{3.3}$$

Next, put $X = -[f_1, \ldots, f_n] \in K[z,s]^{p\times n}$. By Thm. 3.2.c) it is $K[z,s]^p = [X,Q]K[z,s]^{n+p}$, which implies that

$$[X,Q] \text{ is right-invertible over } K[z,s]. \tag{3.4}$$

As for the first statement of the theorem, let us consider the matrices A, B, and C. The choice of the bases for the modules involved imply the relations

$$XA = Q\Pi_-(Q^{-1}sX) \quad \text{and} \quad C = -\Pi_+(Q^{-1}sX),$$

hence

$$X(sI-A) = sX - Q\Pi_-(Q^{-1}sX) = Q(\mathrm{id} - \Pi_-)(Q^{-1}sX) = Q\Pi_+(Q^{-1}sX) = -QC. \tag{3.5}$$

Thus $X = -QC(sI-A)^{-1}$ and Prop. 2.9.b) together with (3.3) and (3.4) yields the desired result.

To prove the coreachability, the following notation is helpful. For an arbitrary matrix $M \in K(z,s)^{\alpha\times\beta}$ let $M^{(\beta_1,\ldots,\beta_p)}_{(\alpha_1,\ldots,\alpha_p)}$ be the $p\times p$-minor given by the columns $\beta_1, \ldots, \beta_p$ and the rows $\alpha_1, \ldots, \alpha_p$.

Equation (3.5) can be written as

$$[X,Q]\begin{bmatrix} sI - A \\ C \end{bmatrix} = 0. \tag{3.6}$$

Regarding this as an equation of matrices over the field $K(z,s)$, one obtains from [11, p. 294] the existence of non-zero coprime elements a, $b \in K[z,s]$ so that

$$[X,Q]^{(i_1,\ldots,i_p)}_{(1,\ldots,p)} = \pm\frac{a}{b}\begin{bmatrix} sI - A \\ C \end{bmatrix}^{(1,\ldots,n)}_{(i_1^*,\ldots,i_n^*)} \tag{3.7}$$

for all $1 \le i_1 < \ldots < i_p \le n+p$ and $1 \le i_1^* < \ldots < i_n^* \le n+p$ with $\{i_1,\ldots,i_p\} \cup \{i_1^*,\ldots,i_n^*\} = \{1,\ldots,n+p\}$. The coprimeness of the full size minors of $[X,Q]$ implies at once $a \in K\backslash\{0\}$. But then $b \in K\backslash\{0\}$ can be deduced from the equation $\deg\det Q = n = \deg\det(sI-A)$ and the monicity of $\det Q$. Thus the full size minors of $\begin{bmatrix} sI-A \\ C \end{bmatrix}$ coincide (up to a minus sign and a non-zero constant) with those of $[X,Q]$ and the result follows from (3.4). □

We illustrate the procedure by the following example.

Example 3.4 Consider again delay-differential equations with commensurate point-delays only. Thus, let $\mathbb{R}[z_1,s]$ act on $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ with $z_1 f(t) = f(t-1)$ and $sf = \dot{f}$. Let

$$[P,Q] = \left[\begin{array}{c|cc} z_1 - 1 & (z_1-1)s^2 & s \\ 0 & (z_1-1)^2 s + s & z_1 - 1 \end{array}\right] \in \mathbb{R}[z_1,s]^{2\times 3}.$$

Then $\det Q = -s^2$ is monic and $Q^{-1}P = -s^{-2}\begin{bmatrix} (z_1-1)^2 \\ -(z_1-1)^3 s - (z_1-1)s \end{bmatrix}$ is strictly proper. From Thm. 3.2.b) we know that

$$S_Q = \operatorname{span}_{\mathbb{R}[z_1]}\left\{\Pi_Q\begin{pmatrix}1\\0\end{pmatrix}, \Pi_Q\begin{pmatrix}0\\1\end{pmatrix}, \Pi_Q\begin{pmatrix}s\\0\end{pmatrix}, \Pi_Q\begin{pmatrix}0\\s\end{pmatrix}\right\}.$$

One calculates

$$\Pi_Q\left(\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \end{bmatrix}\right) = \begin{bmatrix} 1 & (z_1-1)s & -(z_1-1)^2 s & 0 \\ 0 & (z_1-1)^2+1 & -(z_1-1)^3-(z_1-1) & 0 \end{bmatrix}.$$

Therefore, $f_1 := \begin{pmatrix}1\\0\end{pmatrix}$, $f_2 := \begin{pmatrix}(z_1-1)s\\(z_1-1)^2+1\end{pmatrix}$ form a basis of S_Q and

$$\Pi_Q([sf_1,\, sf_2]) = [-(z_1-1)f_2,\, 0],\;\; -P = (1-z_1)f_1,\;\; Q^{-1}[sf_1, sf_2] = \frac{-1}{s}\begin{bmatrix} z_1 - 1 & -s \\ -(z_1-1)^2 s - s & 0 \end{bmatrix}.$$

Use of Thm. 3.3 leads to the first-order system

$$sx = \begin{bmatrix} 0 & 0 \\ 1-z_1 & 0 \end{bmatrix} x + \begin{bmatrix} 1-z_1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \\ (z_1-1)^2+1 & 0 \end{bmatrix} x$$

as a realization for $\ker_{\mathcal{A}}[P,Q]$. This can also be verified by some straightforward calculations.

4 Some results about realizability and minimality

For the realization procedure of the previous section we needed the assumptions that $[P,Q]$ is of full row rank and $\det Q$ is monic, see (3.1). In this final section we will discuss whether this last condition is necessary for realizability of $[P,Q]$ in the sense of Def. 2.7. The answer depends on the specific choice of the systems class, i. e. of the module $\mathcal{A}$ and the operators acting on it. Furthermore, in special cases it will be seen that the Fuhrmann realization is minimal with respect to the dimension of the state module.

Let us first investigate the case $\mathcal{A} = K(z,s)$.

Example 4.1 As we saw in Exp. 2.10.a), for $\mathcal{A} = K(z,s)$ or $\mathcal{A} = K(z)((s^{-1}))$ realizability in the sense of Def. 2.7 implies the existence of a full row rank kernel representation $[P,Q]$ with non-singular Q. From Prop. 2.9.a) we obtain $Q^{-1}P \in K[z][[s^{-1}]]^{p\times m}$. Hence, in this case, the classical realization procedures of i/o-operators over rings apply much better than the Fuhrmann-construction, for an overview see e. g. [1, ch. 4] and [13] and the references therein. In particular, realizability is simply characterized by the property $Q^{-1}P \in K[z][[s^{-1}]]^{p\times m}$ (see [1, Theorems 4.13 and 4.14]) and non-singular left factors of $[P,Q]$ don't matter, i. e.

$$\ker_{\mathcal{A}}[P,Q] = \ker_{\mathcal{A}} U[P,Q] \text{ for all non-singular } U \in K(z,s)^{p\times p}.$$

Hence the monicity of $\det Q$ as needed for the Fuhrmann construction is not necessary for realizability. The following simple example illustrates this.

$$\ker_{\mathcal{A}}\left[\begin{array}{c|cc} z_1 & 0 & -s \\ 0 & z_1 & z_2 \end{array}\right] = \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{A}^3 \;\middle|\; \exists\, x \in \mathcal{A}:\; sx = u,\; y = \begin{bmatrix} -z_2 \\ z_1 \end{bmatrix} x \right\}.$$

Note that the realization is canonical and absolutely minimal in the sense of [20]. One can show by some straightforward calculations that it is not possible to find a polynomial kernel representation $U[P,Q] \in K[z,s]^{2\times 3}$ with a non-singular U such that $\det(UQ)$ is monic and of degree 1.

Let us now switch to the case of delay-differential systems with commensurate delays.

Example 4.2 Choose $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ and $K[z,s] = \mathbb{R}[z_1,s]$ as in Exp. 3.4. It has been sketched in Exp. 2.10.b) that realizability implies $\operatorname{rk}[P,Q] = p$ and, after some normalization, $\det Q$ is monic. Hence the necessity of (3.1). In this special case, one even has a stronger property of the Fuhrmann-realization: it is minimal with respect to the dimension of the state-module (or, in the terminology of behaviors, the number of latent variables). This is a consequence of [8, Prop. 4.3 (iii)]. No results of this type are known for differential systems with noncommensurate delays.

However, even in the case of commensurate delays, the results for systems over fields are not completely generalizable. The following simple example shows that the Fuhrmann realization, although minimal, is not unique up to similarity as a system over whatsoever ring. Let $b \in \mathbb{R}[z_1]$ be arbitrary. The construction of the previous section produces the following realizations

$$\begin{aligned}
\ker_{\mathcal{A}}\left[\begin{array}{c|cc} b(z_1) & s^2 - z_1 + 2 & -1 \\ 0 & 0 & s \end{array}\right] &= \mathcal{B}_{\mathsf{ext}}\left(\begin{bmatrix} 0 & z_1-2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -b(z_1) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0\right) \\
&= \mathcal{B}_{\mathsf{ext}}(A_1,\, B_1,\, C_1,\, 0) \\
\ker_{\mathcal{A}}\left[\begin{array}{c|cc} b(z_1) & s^2 - z_1 + 2 & -z_1 \\ 0 & 0 & s \end{array}\right] &= \mathcal{B}_{\mathsf{ext}}\left(\begin{bmatrix} 0 & z_1-2 & z_1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -b(z_1) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0\right) \\
&= \mathcal{B}_{\mathsf{ext}}(A_2,\, B_2,\, C_2,\, 0)
\end{aligned}$$

Since $\dot{y}_2 = 0$ implies $z_1 y_2 = y_2$, the above behaviors are in fact the same. One can check by straightforward calculations that there exists no non-singular 2×2-matrix T so that

$$(A_2,\, B_2,\, C_2) = (TA_1T^{-1},\, TB_1,\, C_1T^{-1}).$$

Finally we reconsider the class of multidimensional systems studied in [18].

Example 4.3 Let $K[z,s]$ and $\mathcal{A}$ be any of the cases in Exp. 2.3. We will show that equation (2.5) implies for $R = [P, Q]$

i) $\operatorname{rk}[P, Q] = p$,

ii) $\det Q$ is monic,

iii) the Fuhrmann-realization is minimal with respect to the number of latent variables,

hence the requirements of (3.1) are necessary for realizability if one starts with a kernel representation consisting of p equations.
Let (2.5) be valid with all matrices of the sizes as given in Def. 2.7.

i) We show that $\mathcal{B}_{\text{ext}}(A, B, C, D)$ always admits a kernel representation of rank p and with $l \geq p$ equations. Using property (3) of 2.3 this will imply $\operatorname{rk}[P, Q] = p$. To do so, let

$$M := \begin{bmatrix} sI - A & -B \\ 0 & I_m \\ C & D \end{bmatrix}$$

and

$$\ker_{K[z,s]} M^{\mathsf{T}} = \operatorname{im}_{K[z,s]}[\hat{Y}, \hat{P}, \hat{Q}]^{\mathsf{T}} \tag{4.1}$$

with some $[\hat{Y}, \hat{P}, \hat{Q}] \in K[z,s]^{l \times (n+m+p)}$. Then $l \geq p$ and $\operatorname{rk}[\hat{Y}, \hat{P}, \hat{Q}] = p$. Elementary transformations over $K(z,s)$ show even $\operatorname{rk}[\hat{P}, \hat{Q}] = p$. Property (1) of 2.3 yields $\ker_{\mathcal{A}}[\hat{Y}, \hat{P}, \hat{Q}] = \operatorname{im}_{\mathcal{A}} M$ and hence with (2.5)

$$\ker_{\mathcal{A}}[P, Q] = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \ker_{\mathcal{A}}[sI - A, -B] = \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{A}^{m+p} \;\middle|\; \begin{pmatrix} 0 \\ u \\ y \end{pmatrix} \in \operatorname{im}_{\mathcal{A}} M \right\} = \ker_{\mathcal{A}}[\hat{P}, \hat{Q}].$$

Now (3) of 2.3 implies

$$[\hat{P}, \hat{Q}] = W[P, Q] \text{ for some } W \in K[z,s]^{l \times p}, \tag{4.2}$$

and therefore $\operatorname{rk}[P, Q] = p$.

ii) & iii) First we show that

$$\ker_{K[z,s]} M^{\mathsf{T}} = \operatorname{im}_{K[z,s]}[Y, P, Q]^{\mathsf{T}} \tag{4.3}$$

with some $Y \in K[z,s]^{p \times n}$; that is, $\ker_{K[z,s]} M^{\mathsf{T}}$ is a free module. This will allow, similarly to the proof of Thm. 3.3, to establish monicity of $\det Q$ and minimality of the realization. Equation (4.3), of course, is a special situation due to the sizes of the matrices involved. To establish (4.3), we use Prop. 2.9.a) and obtain the non-singularity of Q as well as $-Q^{-1}P = C(sI - A)^{-1}B + D$. Moreover, choosing $u = 0$ in (2.5) shows that $\ker_{\mathcal{A}}(sI - A) \subseteq \ker_{\mathcal{A}} QC$. Therefore, again, (3) of 2.3 leads to a matrix $Y \in K[z,s]^{p \times n}$ such that $Y(sI - A) = -QC$. This altogether results in the matrix equation $[Y, P, Q]M = 0$, hence "$\supseteq$" of (4.3). As for "$\subseteq$" let $a \in K[z,s]^{n+m+p}$ with $M^{\mathsf{T}} a = 0$. Then (4.1) and (4.2) yield $a = [\hat{Y}, \hat{P}, \hat{Q}]^{\mathsf{T}} v = [Y, P, Q]^{\mathsf{T}} W^{\mathsf{T}} v \in \operatorname{im}_{K[z,s]}[Y, P, Q]^{\mathsf{T}}$ for some $v \in K[z,s]^l$.
From (4.3) we may conclude that $[Y, P, Q]$ is minor-prime, see [23, Thm. 3.3.8], that is, the full size minors are coprime in $K[z,s]$. Along the same line of arguments as in (3.6) and (3.7), one obtains from $[Y, P, Q]M = 0$ a relation $b \det Q = \det(sI - A)$ with some $b \in K[z,s]$. Thus, $\det Q$ is monic and the dimension of each realization is at least $\deg \det Q$. This shows ii) and iii).

5 Conclusion

In this paper we showed that the polynomial model of Fuhrmann provides a first-order behavioral realization for quite a general class of multi-operator systems. Only in special cases the minimality of the Fuhrmann realization can be proven. Even in these cases the question about uniqueness of minimal realizations appears to be quite difficult to answer.

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