MA111: Contemporary mathematics

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Schedule:

- Participation quiz on BB should be done **today** (and take like 30 seconds)
- HW 10.6 EZ is due Today, Sep 21st, 2011.
- Exam 2 is Monday, Oct 3rd, during class.

Today we will look at borrowing money for several years, 10.6, amortized loans.

10.6 EZ: Review of short installment loans

- Two key ideas:
- Payments not only lower the debt, they lower the interest too
  Payments basically **earn** interest
- Moving from future value to present value is just dividing by $1 + p$
  Fancy formula is going to call it multiplying by $q = \frac{1}{1+p}$
- With just a few installments, we calculate by hand
- With 20 or 30 or 360, we need a formula

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- How long does it take to pay it off? almost 6 months

$$\begin{aligned}
\$1000.00 &\xrightarrow{\text{plus 2\% minus \$200}} \\
\$\ 820.00 &\xrightarrow{\text{plus 2\% minus \$200}} \\
\$\ 636.40 &\xrightarrow{\text{plus 2\% minus \$200}} \\
\$\ 449.13 &\xrightarrow{\text{plus 2\% minus \$200}} \\
\$\ 258.11 &\xrightarrow{\text{plus 2\% minus \$200}} \\
\$\ \ 63.27 &\xrightarrow{\text{plus 2\% minus \$64.54}} \\
\$\ \ \ 0.00
\end{aligned}$$

10.6 EZ: The very short mortgage / credit card

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$$Mq + Mq^2 + Mq^3 = Mq\frac{1-q^3}{1-q}$$

10.4: Adding up numbers!

- A frog jumps halfway to the end of the log:

$$d = \frac{1}{2}$$

- He does it again, but literally:

$$d = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

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- If he keeps doing this, how far does he get? $\frac{15}{16}, \frac{31}{32}, \ldots, 1$?

10.4: Frog math

- Suppose Robin (the frog) is jumping too, but only "half" as far
- Robin jumps a quarter of the way, and then a quarter of that, and then a quarter of that, etc.
- How far does Robin make it?
- (Prepare to present your answer at the board)

- What if he jumped 8 times? Just add them up...

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- Kind of a pain. Crazy trick: multiply by $(1 - q)$,
  take the original, and subtract q times the original from it:

$$\begin{array}{rcccccccc} & q^1 + & q^2 + & q^3 + & q^4 + & q^5 + & q^6 + & q^7 + & q^8 \\ - & q^2 - & q^3 - & q^4 - & q^5 - & q^6 - & q^7 - & q^8 - & q^9 \\ \hline \end{array}$$

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- $(1 - q)(q + q^2 + \cdots + q^8) = q - q^9 = q(1 - q^8)$

$$q + q^2 + \cdots + q^8 = q\frac{1 - q^8}{1 - q}$$

10.4: Factoring froggies

- Difference of squares: $1 - q^2 = (1 - q)(1 + q)$
- Difference of cubes: $1 - q^3 = (1 - q)(1 + q + q^2)$
- Difference of fourths: $1 - q^4 = (1 - q)(1 + q + q^2 + q^3)$
- Difference of fifths: $1 - q^5 = (1 - q)(1 + q + q^2 + q^3 + q^4)$
- Difference of 360ths: $1 - q^{360} = (1 - q)(1 + q + \cdots + q^{358} + q^{359})$

10.6: The formula

- Difference of 360ths

$$1 - q^{360} = (1 - q)(1 + q + \cdots + q^{358} + q^{359})$$

- Multiply by q

$$q(1 - q^{360}) = (1 - q)(q + q^2 + \cdots + q^{359} + q^{360})$$

- Divide by $1 - q$

$$q\frac{1 - q^{360}}{1 - q} = q + q^2 + \cdots + q^{359} + q^{360}$$

- Multiply by M

$$Mq\frac{1 - q^{360}}{1 - q} = Mq + Mq^2 + \cdots + Mq^{360}$$

10.6: Using the formula

- For some reason you charge \$5000 on your credit card
- Realizing the error of your mistake, you swear never to spend on that card again
- You make monthly payments of \$500 on it, with 35% APR compounded monthly
- How does that work out for you? \$5000/\$500 = 10, should be 10 months, eh?

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$$\begin{aligned} M &= \$500 \\ q &= 1/(1 + 0.35/12) \\ T &= 12 \\ P &= Mq\frac{1-q^{12}}{1-q} = \$5001.85 \end{aligned}$$

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- Takes a **year**, not 10 months. Where did the extra \$1000 go?

10.6: It's a false economy

- Why not save yourself money by making a smaller payment? \$200 should do it.
- \$5000/\$200 = 25 months, just a little over 2 years, no biggy
- How much of a loan would 3 years and 9 months of payments cover?

$$\begin{aligned} M &= \$200 \\ q &= 1/(1 + 0.35/12) \\ T &= 45 \\ P &= Mq\frac{1-q^{45}}{1-q} = \$4976.59 \end{aligned}$$

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- Takes over 45 months to pay it back, where did the extra (20 months)(\$200 per month) = \$4000 go?

Homework

- Calculations using formula: installment loans (what happens), installment loans (calculating the payment)
- Participation (15%): There is a quiz on blackboard, under **Assignments**. Should do it today. Due by Sunday.
- Read section 10.6 of the textbook. Skim 10.4 - 10.5.
- Online homework (30%):
  - HW 10.6 EZ is due Today.
  - HW 10.6 is due Monday.