Part A: Perceived Value

1. Given a description of the perceived value, calculate some specific numbers. Andy, Bill, Chad, Dirk, and Evan find the dread pirates Alex, Bart, and Carl cutlassed to death next to a pile of treasure containing a disgruntled parrot, some dirty underwear, some silver dollars, and a slightly sandy cake. Andy's perception of the loot's value is in the table to the right.

Per. Value	Andy
Parrot	\$10
Underoos	\$1
100 dollars	\$100
Cake	\$24
Total	\$135

(a) How much does Andy think a portion including the parrot and half the cake is worth?

The parrot is worth \$10 to Andy, and half the cake is worth $(\frac{1}{2})(\$24) = \$12$, so that is $\$10 + \$12 = \$22$ total.

(b) How much does Andy think a portion including the silver dollars and the underwear is worth?

The silver dollars are worth \$100 to Andy, and the underwear is worth \$1, so that is $\$100 + \$1 = \$101$ total.

2. Determine the value of a fair share.

(a) If Bill thinks the entire pile of loot is worth \$120 but he jointly owns it with 5 people (including himself), how much does he consider his (minimum) fair share?

5 people jointly own \$120, so each owns $\frac{\$120}{5} = \$24$. Bill considers \$24 to be (exactly) a fair share.

(b) (From this point on, Dirk and Evan have cutlassed each other and are no longer joint owners). If Bill thinks the entire pile of loot is worth \$120 but he jointly owns it with 3 people, how much does he consider his (minimum) fair share?

With only 3 surviving, each owns a third, so Bill considers $\frac{\$120}{3} = \$40$ to be his fair share.

(c) Andy, Bill, and Chad evenly split the cost of a \$36 cake. How much does Bill consider his (minimum) fair share?

$\frac{\$36}{3} = \$12$

3. Would Andy consider the silver dollars and the underwear (at least) a fair share of the pirate loot? (Assuming Bill and Chad can avoid cutlassing each other).

Yes! The silver dollars and the underwear are worth \$101 to him, while the entire pile is worth \$135, so that is $\frac{\$101}{\$135} \approx 75\%$ of the loot. His fair share is only $\frac{\$135}{3} = \$45$, that is, 33%.

Part B: Lone Divider

1. Describe what happens in Steinhaus's method of the Lone Divider.
   (a) A cake is to be divided using Steinhaus's method of the Lone Divider. Andy splits the cake into three pieces, X, Y, and Z. Bill commits to his approval of X and Z, Chad commits to his approval of Z. What happens now?

   Chad gets Z, Bill gets X, and Andy gets the leftover: Y. Since Bill approved of at least 2 of the pieces, there was no fight.

   (b) A pie is to be divided using Steinhaus's method of the Lone Divider. Andy splits the pie into three pieces, X, Y, and Z. Bill commits to his approval of Z, and Chad commits to his approval of Z. What happens now?

   Bill and Chad both want Z, so they give Andy either X or Y (say X) and then split the other and Z (so Bill and Chad split Y and Z). They can use I-Cut-You-Choose to divide the combined piece.

   (c) A cookie is to be divided using Steinhaus's method of the Lone Divider. Andy splits the cookie into three pieces, X, Y, and Z. Bill commits to his approval of Z, and Chad commits to his approval of Y. What happens now?

   Bill gets Z, Chad gets Y, and Andy gets the leftover X. Even though Bill and Chad only approved of one piece, they approved of different pieces so there is no fight.

2. Now use the information on the perceived value to find the problem in a suggested strategy.
   (a) A bowl of mashed potatoes is to be divided using Steinhaus's method of the Lone Divider. Andy scoops the potatoes into three lumps, X, Y, and Z. Bill thinks X is worth \$6, Y is worth \$5, Z is worth \$1. Bill is considering approving only of X. What could Chad do that would ruin Bill's plan?

   Chad might only approve of X, forcing a tie-breaker. Andy might get Y, leaving Bill and Chad to split X and Z, a total of \$6+\$1. Bill might only get his fair share of \$7, that is, Bill might get stuck with \$3.50, way less than the \$5 he could have guaranteed himself, and even less than the \$4 he considered his fair share.

   (b) A glass of orange juice is to be divided using Steinhaus's method of the Lone Divider. Andy pours the OJ into three glasses, X, Y, and Z. Andy thinks X is worth \$5, Y is worth \$4, and Z is worth \$3. What could Bill do that would ruin Andy's plan?

   Bill might approve only X, and Chad might approve only Y, leaving Andy with Z. Andy would get \$3, which is less than his fair share, \$4.

3. Now give a winning strategy for each player:

(a) A bowl of mashed potatoes is to be divided using Steinhaus's method of the Lone Divider. Andy scoops the potatoes into three lumps, X, Y, and Z. Bill thinks X is worth \$6, Y is worth \$5, Z is worth \$1. What should Bill do now?

Bill should the play the winning strategy: honesty. He should approve of both X and Y, guaranteeing him a share of at least \$5.

(b) A glass of orange juice is to be divided using Steinhaus's method of the Lone Divider. Andy needs to pour the OJ into three glasses, X, Y, and Z. How much should Andy put in each glass?

Andy should play the winning strategy: honesty. He should divide the OJ evenly into the three glasses, each having $\frac{\$5+\$4+\$3}{3} = \$4$ of OJ in it, according to Andy's perception of the value at least.

Part C: Sealed bids

1. Describe what happens in Knaster's method of the Sealed Bids.

   (a) A glass elephant is to be divided using Knaster's method of the Sealed Bids. The bids are: Andy \$10; Bill \$8; Chad \$6. What happens now?

   In step 2, Andy trades the \$10 for the Elephant. He used to own all his \$10 and a third of the elephant, but now he owns all the elephant and only a third of his \$10. In step 3, Andy gets back his fair share of his bid, $\frac{\$10}{3}$; Bill gets back his fair share of his bid $\frac{\$8}{3}$; and Chad gets back his fair share of his bid $\frac{\$6}{3}$. This leaves

   $$\$10 - \frac{\$10}{3} - \frac{\$8}{3} - \frac{\$6}{3} = \$10 - \frac{\$10 + \$8 + \$6}{3} = \$10 - \frac{\$24}{3} = \$10 - \$8 = \$2$$

   left over, which, in step 4, is split evenly amongst the three owners: Andy gets another $\frac{\$2}{3}$, Bill gets another $\frac{\$2}{3}$, and Chad gets another $\frac{\$2}{3}$.

   Andy is the high bidder, so gets the loot. Andy pays Bill $\frac{\$10-\$8+\$8}{3} = \frac{\$10}{3}$, and Andy pays Chad $\frac{\$10-\$8+\$6}{3} = \frac{\$8}{3}$. In general, the winner gets the loot and pays each of the losers $(W-A+B)/n$ where W is the winning bid, A is the average bid, B is the loser's own bid, and n is the number of players, including the winner.

   (b) A crystal dove is to be divided using Knaster's method of the Sealed Bids. The bids are: Andy \$10; Bill \$10; Chad \$10. If we decide to give the dove to Andy, what happens as far as the money is concerned?

   Andy pays Bill and Chad both $(\$10+\$10-\$10)/3 = \$3.33$, and Andy gets to keep the dove.

   (c) A saltine cracker is to be divided using Knaster's method of the Sealed Bids. The bids are: Andy \$0.10; Bill \$0.25; Chad \$0.00. What happens now?

   $W = \$0.25$ and $A = \$0.12$, so Bill gets the saltine cracker, and then Bill pays Andy $(\$0.10 + \$0.25 - \$0.12)/3 = \$0.08$, and Bill pays Chad $(\$0.00 + \$0.25 - \$0.12)/3 = \$0.04$.

2. Now use the information on the perceived value to find the problem in a suggested strategy.

   (a) A Fabergé peanut shell is to be divided using Knaster's method of the Sealed Bids. Andy thinks it is worth \$100, but is considering bidding \$90. What can go wrong?

   Everyone else might bid \$93. Andy will get \$30 back in Step 3, and in step 4, he will get another \$0.33, totalling \$30.33. However, his fair share of the \$100 shell is \$33.33, so he managed to lose himself \$3.

(b) A Fabergé peach pit is to be divided using Knaster's method of the Sealed Bids. Bill thinks it is worth \$100, but is considering bidding \$110. What can go wrong?

Everyone else might bid \$108. Bill will win the peach pit, but end up paying \$108/3 = \$36 to Andy and Chad in step 3, and then another \$0.22 in step 4. Instead of getting his fair share of \$100/3 = \$33.33, he'll get a \$100 egg for \$72.44, meaning he only got \$27.66 worth of egg net.

3. Now give a winning strategy for each player:

(a) A Fabergé peanut shell is to be divided using Knaster's method of the Sealed Bids. Andy thinks it is worth \$100. What is a winning strategy for Andy?

Andy should play the winning strategy: honesty. Andy should bid \$100.

(b) A Fabergé peach pit is to be divided using Knaster's method of the Sealed Bids. Bill thinks it is worth \$100. What is a winning strategy for Bill?

Bill should play the winning strategy: honesty. Bill should bid \$100.

Part D: Last Diminisher

1. Describe what happens in Banach-Knaster's method of the Last Diminisher.
   (a) A desert island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy indicates a portion of the island, Bill approves of that portion, Chad says it is too big and shrinks it a bit, Dave says the result is too small. What happens now?

   > Chad was the last to say "Keep" or "Shrink", so Chad gets his portion of the island, and Bill, Chad, and Dave split the remaining portion of the desert island.

   (b) An appetizer island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy indicates a portion of the island, Bill thinks it is too small, Chad thinks it is too small, and Dave thinks it is too small. What happens now?

   > Andy was the last to say "Keep" or "Shrink", so Andy gets his portion of the island, and Bill, Chad, and Dave split the remaining portion of the appetizer island.

   (c) A breakfast island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy indicates a portion of the island, Bill thinks it is just right, Chad thinks it is just right, and Dave thinks it is too small. What happens now?

   > In summary, Andy: Shrink; Bill: Keep; Chad: Keep; Dave: Pass. This means Chad gets his portion of the island, since he was the last to say "Keep" or "Shrink". Andy, Bill, and Dave now split the remaining portion of the breakfast island.

2. Now use the information on the perceived value to find the problem in a suggested strategy for four players (Andy, Bill, Chad, and Dave).
   (a) A tropical island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy indicates a portion of the island, Bill thinks it is too small, Chad thinks it is worth about 50% of the total value of the island, and is considering saying "it is just right". What can go wrong?

   > Dave might say "Keep" instead of "Pass", so that Dave gets the big piece instead. This leaves only half of the island left for Andy, Bill, and Chad, so that Chad can only guarantee a third of a half (better known as a sixth) of the island for himself.

   (b) An arctic island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy is considering indicating 20% of the island as his portion. What can go wrong?

   > Bill, Chad, and Dave might all Pass, leaving Andy with the small piece. Andy's fair share is 25% (a quarter), and so he would feel cheated by getting only 20%.

3. Now give a winning strategy for each player:

(a) A tropical island is to be divided using Banach-Knaster's method of the Last Diminisher. Andy indicates a portion of the island, Bill thinks it is too small, Chad thinks it is worth about 50% of the total value of the island. What should Chad do in order to guarantee winning?

> Chad should use the winning strategy: honesty. He should shrink his portion down to exactly 1/4 of the island (by value, according to his perception of the value). Any larger and he risks someone else getting the "big piece", and any smaller and he risks getting the "small piece" himself.

(b) An arctic island is to be divided using Banach-Knaster's method of the Last Diminisher. How much of the island (by value) should Andy indicate in his portion?

> Andy should use the winning strategy: honesty. He should shrink his portion down to exactly 1/4 of the island (by value, according to his perception of the value). Any larger and he risks someone else getting the "big piece", and any smaller and he risks getting the "small piece" himself.