MA322-001 May 2 Practice Exam Name: Key

1. Spectral decomposition. Explain your answers. The following vectors are used this page:

$$\vec{\mathbf{u}} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \vec{\mathbf{v}} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{\mathbf{w}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \cdot \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

(a) Compute $A = \vec{\mathbf{u}} \cdot 2 \cdot \vec{\mathbf{u}}^T + \vec{\mathbf{v}} \cdot 4 \cdot \vec{\mathbf{v}}^T + \vec{\mathbf{w}} \cdot 6 \cdot \vec{\mathbf{w}}^T$

$$A = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

(b) Find 3 linearly independent eigenpairs of A.

$$A\vec{u} = (\vec{u} \cdot 2 \cdot \vec{u}^T + \vec{v} \cdot 4 \cdot \vec{v}^T + \vec{w} \cdot 6 \cdot \vec{w}^T)\vec{u}$$

$$= \vec{u} \cdot 2 \cdot \vec{u}^T\vec{u} + \vec{v} \cdot 4 \cdot \vec{v}^T\vec{u} + \vec{w} \cdot 6 \cdot \vec{w}^T\vec{u}$$

$$= \vec{u} \cdot 2 \cdot 1 + \vec{v} \cdot 4 \cdot 0 + \vec{w} \cdot 6 \cdot 0$$

$$= \vec{u} \cdot 2$$

so $(2, \vec{u})$. By the same reasoning $(4, \vec{v})$ and $(6, \vec{w})$

Bonus: Find a 4th linearly independent eigenpair (or just explain how).

2. Quadratic forms

(a) Compute a matrix B so that:

$$Q(\vec{\mathbf{x}}) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 2(x_1+x_2)^2 + (x_1+x_3)^2 + (x_2+x_4)^2 + 2(x_3+x_4)^2$$

$$Q(\vec{x}) = 2(x_1^2 + 2x_1x_2 + x_2^2) + (x_1^2 + 2x_1x_3 + x_3^2) + \cdots$$

$$= 2x_1^2 + 4x_1x_2 + 2x_2^2 + x_1^2 + 2x_1x_3 + x_3^2 + x_2^2 + 2x_2x_4 + \cdots$$

$$= 3x_1^2 + 4x_1x_2 + 2x_1x_3 + 0x_1x_4$$
$$+ 3x_2^2 + 0x_2x_3 + 2x_2x_4$$
$$+ 3x_3^2 + 4x_3x_4$$
$$+ 3x_4^2$$

$$B = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

(b) Suppose B had eigenpairs $(0, \vec{\mathbf{t}})$, $(2, \vec{\mathbf{u}})$, $(4, \vec{\mathbf{v}})$, $(6, \vec{\mathbf{w}})$. Explain how to find the values of x_i so that $Q(\vec{\mathbf{x}})$ is maximized subject to $\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = 1$.

If $\vec{t}, \vec{u}, \vec{v}, \vec{w}$ are orthonormal, then $\vec{w}$ is the best direction so just set $\vec{x} = \vec{w}$ to get

$$Q(\vec{x}) = \vec{x}^T B \vec{x} = \vec{w}^T (6\vec{w}) = 6\vec{w}^T\vec{w} = 6 \cdot 1 = 6$$

Bonus: Actually find the values of x_i.

3. Diagonal SVD. Let $C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} f \\ g \\ h \\ i \end{bmatrix}$

(a) Find an expression for $\vec{\mathbf{x}}$ (in terms of f, g, h, i) so that $\|C\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ is as small as possible.

$\vec{x} = \begin{bmatrix} f/2 \\ g/3 \\ h/4 \end{bmatrix}$ because

$$C\vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f/2 \\ g/3 \\ h/4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} (f/2) + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} (g/3) + \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} (h/4)$$

$C\vec{x} = \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix}$ ← always 0, cannot fix

$$= \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ g \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h \\ 0 \end{bmatrix}$$

$= \begin{bmatrix} f \\ g \\ h \\ 0 \end{bmatrix}$ is as close as possible

(b) Find a value for $\vec{\mathbf{x}}$ so that $\|C\vec{\mathbf{x}}\|$ is as large as possible subject to $\|\vec{\mathbf{x}}\| \leq 1$.

$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad \|C\vec{x}\| = \left\| \begin{bmatrix} 2a \\ 3b \\ 4c \end{bmatrix} \right\|$

$= \sqrt{4a^2 + 9b^2 + 16c^2}$

Just Like 2(b) for $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

take $a=0,\ b=0,\ c=1$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\|C\vec{x}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\| = \sqrt{16} = 4$

4. Let $\vec{\mathbf{u}}_1 = \begin{bmatrix} 0.50 \\ -0.50 \\ -0.50 \\ 0.50 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 0.50 \\ -0.50 \\ 0.50 \\ -0.50 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 0.36 \\ -0.48 \\ -0.80 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} -0.48 \\ 0.64 \\ -0.60 \end{bmatrix}$.

Set $D = \vec{\mathbf{u}}_1 \cdot 10 \cdot \vec{\mathbf{v}}_1^T + \vec{\mathbf{u}}_2 \cdot 15 \cdot \vec{\mathbf{v}}_2^T = \begin{bmatrix} -1.80 & 2.40 & -8.50 \\ 1.80 & -2.40 & 8.50 \\ -5.40 & 7.20 & -0.50 \\ 5.40 & -7.20 & 0.50 \end{bmatrix}$ and $\vec{\mathbf{d}} = \vec{\mathbf{u}}_1 \cdot 2 + \vec{\mathbf{u}}_2 \cdot 4 = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}$.

(a) Find $\vec{\mathbf{x}}$ so that $\|D\vec{\mathbf{x}} - \vec{\mathbf{d}}\|$ is as small as possible.

Let $\vec{x} = a\vec{v}_1 + b\vec{v}_2$

$$D\vec{x} = (\vec{u}_1 10 \vec{v}_1^T + \vec{u}_2 15 \vec{v}_2^T)(a\vec{v}_1 + b\vec{v}_2)$$

$$= \vec{u}_1 10 \vec{v}_1^T a \vec{v}_1 + \vec{u}_2 15 \vec{v}_2^T b \vec{v}_2$$

$$= \vec{u}_1 (10a) + \vec{u}_2 (15b)$$

$$\vec{d} = \vec{u}_1 (2) + \vec{u}_2 (4)$$

So $a = \frac{1}{5}$ $\quad b = \frac{4}{15}$

$$\vec{x} = \frac{1}{5}\vec{v}_1 + \frac{4}{15}\vec{v}_2$$

(b) Find $\vec{\mathbf{x}}$ so that $\|D\vec{\mathbf{x}}\|$ is as large as possible, subject to $\|\vec{\mathbf{x}}\| \leq 1$.

Only the 15 is efficient. Want $D\vec{x} = \vec{u}_2 \cdot 15$

so take $\vec{x} = \vec{v}_2$