MA322-001 May 2 Practice Exam Name: ______________________

1. Spectral decomposition. Explain your answers. The following vectors are used this page:

$$\vec{\mathbf{u}} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{\mathbf{w}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

(a) Compute $A = \vec{\mathbf{u}} \cdot 2 \cdot \vec{\mathbf{u}}^T + \vec{\mathbf{v}} \cdot 4 \cdot \vec{\mathbf{v}}^T + \vec{\mathbf{w}} \cdot 6 \cdot \vec{\mathbf{w}}^T$

(b) Find 3 linearly independent eigenpairs of A.

Bonus: Find a 4th linearly independent eigenpair (or just explain how).

2. Quadratic forms

(a) Compute a matrix B so that:

$$Q(\vec{\mathbf{x}}) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 2(x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_4)^2 + 2(x_3 + x_4)^2$$

(b) Suppose B had eigenpairs $(0, \vec{\mathbf{t}})$, $(2, \vec{\mathbf{u}})$, $(4, \vec{\mathbf{v}})$, $(6, \vec{\mathbf{w}})$. Explain how to find the values of x_i so that $Q(\vec{\mathbf{x}})$ is maximized subject to $\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = 1$.

Bonus: Actually find the values of x_i.

3. Diagonal SVD. Let $C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} f \\ g \\ h \\ i \end{bmatrix}$

(a) Find an expression for $\vec{\mathbf{x}}$ (in terms of f, g, h, i) so that $\|C\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ is as small as possible.

(b) Find a value for $\vec{\mathbf{x}}$ so that $\|C\vec{\mathbf{x}}\|$ is as large as possible subject to $\|\vec{\mathbf{x}}\| \leq 1$.

4. Let $\vec{\mathbf{u}}_1 = \begin{bmatrix} 0.50 \\ -0.50 \\ -0.50 \\ 0.50 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 0.50 \\ -0.50 \\ 0.50 \\ -0.50 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 0.36 \\ -0.48 \\ -0.80 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} -0.48 \\ 0.64 \\ -0.60 \end{bmatrix}$.

Set $D = \vec{\mathbf{u}}_1 \cdot 10 \cdot \vec{\mathbf{v}}_1^T + \vec{\mathbf{u}}_2 \cdot 15 \cdot \vec{\mathbf{v}}_2^T = \begin{bmatrix} -1.80 & 2.40 & -8.50 \\ 1.80 & -2.40 & 8.50 \\ -5.40 & 7.20 & -0.50 \\ 5.40 & -7.20 & 0.50 \end{bmatrix}$ and $\vec{\mathbf{d}} = \vec{\mathbf{u}}_1 \cdot 2 + \vec{\mathbf{u}}_2 \cdot 4 = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}$.

(a) Find $\vec{\mathbf{x}}$ so that $\|D\vec{\mathbf{x}} - \vec{\mathbf{d}}\|$ is as small as possible.

(b) Find $\vec{\mathbf{x}}$ so that $\|D\vec{\mathbf{x}}\|$ is as large as possible, subject to $\|\vec{\mathbf{x}}\| \leq 1$.