1.8: Matrices are functions

An $R \times C$ matrix A with R rows and C columns is also a function whose domain is vectors $\vec{\mathbf{c}}, \vec{\mathbf{x}}$ of size C and whose codomain is vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_C, \vec{\mathbf{b}}$ of size R. For example $A\vec{\mathbf{c}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \ldots + c_C\vec{\mathbf{v}}_C$ expresses the output of $A(\vec{\mathbf{c}})$ as a linear combination of the vectors $\vec{\mathbf{v}}_i$ that are the columns of A.

Not all functions from vectors to vectors are matrices: only the ones that satisfy the two axioms: $A(r\vec{\mathbf{c}}) = r \cdot A(\vec{\mathbf{c}})$ and $A(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = A(\vec{\mathbf{x}}) + A(\vec{\mathbf{y}})$ for all vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in the domain and all scalars r.

1.9: Blackbox matrix

The vectors $\vec{\mathbf{e}}_i$ that have a 1 in the ith position and a 0 elsewhere are called the **standard basis vectors**. If $A(\vec{\mathbf{e}}_i) = \vec{\mathbf{v}}_i$, then $\vec{\mathbf{v}}_i$ is exactly the ith column of A. In order to find the columns of A when A is only described in words, just calculate $A(\vec{\mathbf{e}}_i)$, the image of each standard basis vector under the action of A. By the two matrix axioms, if $\vec{\mathbf{x}} = x_1\vec{\mathbf{e}}_1 + \ldots + x_n\vec{\mathbf{e}}_n$, then $A(\vec{\mathbf{x}}) = x_1\vec{\mathbf{v}}_1 + \ldots + x_n\vec{\mathbf{v}}_n$.

2.1: Matrix operations

If we can take linear combinations of elements of the codomains, we can take linear combinations of the functions. If A, B are matrices of the same size and r is a scalar, then define $A+B$ to be the matrix of the same size that takes $\vec{\mathbf{x}}$ to $A(\vec{\mathbf{x}})+B(\vec{\mathbf{x}})$ and rA to be the matrix of the same size that takes $\vec{\mathbf{x}}$ to $r(A(\vec{\mathbf{x}}))$. We can check that this makes $A+B$ and rA into matrices.

Example 1,2: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$.

$A + B$

$A + C$

$2B$

$A - 2B$

2.1.b: Matrices are vectors

We've seen different types of vectors. 2D vectors. 3D vectors. Chemical vectors. Each type of vector obeys the same basic rules: you can add and subtract vectors and multiply them by numbers. If $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are vectors of the same type and r, s are numbers, then (1) $(r+s)\vec{\mathbf{a}} = r\vec{\mathbf{a}} + s\vec{\mathbf{a}}$, (2) $r(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = r\vec{\mathbf{a}} + r\vec{\mathbf{b}}$, (3) $1\vec{\mathbf{a}} = \vec{\mathbf{a}}$, (4) $r(s\vec{\mathbf{a}}) = (rs)\vec{\mathbf{a}}$, (5) $0\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}}$, and (6) $\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$. We abbreviate $0\vec{\mathbf{a}}$ as $\vec{\mathbf{0}}$ and $(-1)\vec{\mathbf{a}}$ as $-\vec{\mathbf{a}}$, and we write $\vec{\mathbf{b}} + (-1)\vec{\mathbf{a}}$ as $\vec{\mathbf{b}} - \vec{\mathbf{a}}$.

Now we have a new type of vector for every pair of positive integers R, C: the $R \times C$ matrices are vectors where we can form linear combinations.

2.1.c: Matrix multiplication We define the composition of two matrices AB by the rule $(AB)\vec{\mathbf{x}} = A(B\vec{\mathbf{x}})$. This requires that codomain of B be the domain of A, that is, if A is $m \times n$ and B is $n' \times p$, then we must have $n = n'$.

Suppose B has columns $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_p$ all of which are in the domain of A. Then AB has columns $A\vec{\mathbf{b}}_1, A\vec{\mathbf{b}}_2, \ldots, A\vec{\mathbf{b}}_p$. In particular, AB has the same number of columns as B does.

$$\text{If } B = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{b}}_1 & \vec{\mathbf{b}}_2 & \ldots & \vec{\mathbf{b}}_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \text{ then } AB = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ A\vec{\mathbf{b}}_1 & A\vec{\mathbf{b}}_2 & \ldots & A\vec{\mathbf{b}}_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\text{If } \vec{\mathbf{b}}_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1n} \end{bmatrix} \text{ and } A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \ldots & \vec{\mathbf{a}}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \text{ then } A\vec{\mathbf{b}}_1 = b_{11}\vec{\mathbf{a}}_1 + b_{12}\vec{\mathbf{a}}_2 + \ldots + b_{1n}\vec{\mathbf{a}}_n.$$

These satisfy the additional axioms similar to numbers (most of which follow from being functions that output vectors): $A(BC) = (AB)C$, $(A + B)C = AC + BC$, $A(B + C) = AB + AC$, $x(AB) = (xA)B = A(xB)$. Additionally the matrix I_m consisting of the m different m-D standard basis vectors acts like 1, $I_m A = A = AI_n$ if A is $m \times n$.

Example 3: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

AB BA

Example 7: Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

AB BA

See exercise 10 and 12 for more extreme examples.

2.1.d: Square matrices See exercise 11 for an important square example. Also I_n.

2.1.e: Transpose See exercise 27 for an important construction.

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HW1.9 #1 If $A(\vec{\mathbf{e}}_1) = (3, 1, 3, 1)$ and $A(\vec{\mathbf{e}}_2) = (-5, 2, 0, 0)$, then what is the matrix of A, assuming A is a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^4$ and $\vec{\mathbf{e}}_1 = (1, 0)$ and $\vec{\mathbf{e}}_2 = (0, 1)$.

HW1.9 #15 Find the matrix of A where $A(x, y, z) = (2x - 4y, x - z, -y + 3z)$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$.

HW1.9 #17 Find the matrix of A where $A(a, b, c, d) = (a + 2b, 0, 2b + d, b - d)$ and $A : \mathbb{R}^4 \to \mathbb{R}^4$.

2.1 (#11a) Find the matrix of D where $D(x, y, z) = (5x, 3y, 2z)$.

2.1 (#11b) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, find the matrix of DA.