MA322-007 Mar 12 Practice Exam Name:______________________

1. Are these subspaces? Show the subspace check.

(a) $\{(x,y) \in \mathbb{R}^2 : y = (x-1)^2 - 1\} = W$

✓(0) $\vec{0} = (0,0)$ $x = y = 0$, $0 = (0-1)^2 - 1 = 0$

(+) $\vec{u} = (1,-1)$, $\vec{v} = (2,0)$ are in W
but $\vec{u} + \vec{v} = (3,-1)$ has $(3-1)^2 - 1 = 4 - 1 = 3 \neq -1$

Not a subspace

(b) $\left\{ \begin{bmatrix} a-2b \\ b-2a \\ a+b \end{bmatrix} : a, b \in \mathbb{R} \right\} = W$

All checks are satisfied so this is a subspace.

✓(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ If $a = b = 0$, $\begin{bmatrix} a-2b \\ b-2a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

✓(+) $\begin{bmatrix} e-2f \\ f-2e \\ e+f \end{bmatrix} + \begin{bmatrix} c-2d \\ d-2c \\ c+d \end{bmatrix} = \begin{bmatrix} (e+c)-2(f+d) \\ (f+d)-2(e+c) \\ (e+c)+(f+d) \end{bmatrix}$ which is in W $\left(\begin{matrix} a = e+c \\ b = f+d \end{matrix} \right)$

✓($\cdot$) $c \begin{bmatrix} d-2e \\ e-2d \\ d+e \end{bmatrix} = \begin{bmatrix} cd-2ce \\ ce-2cd \\ cd+ce \end{bmatrix}$ which is in W $\left(\begin{matrix} a = cd \\ b = ce \end{matrix} \right)$

(c) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^2 : x + y + z = 2x + 3y = 0 \right\} = W$

All checks are satisfied so this is a subspace.

✓(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ If $x = y = z = 0$ the equations are satisfied

✓(+) Choose from W: $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\vec{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$

$a+d+b+e+c+f$
$= (a+b+c) + (d+e+f) = 0 + 0 = 0$ ✓
$2(a+d) + 3(b+e) = (2a+3b) + (2d+3e) = 0 + 0 = 0$

✓($\cdot$) Choose from W:
$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $d \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} da \\ db \\ dc \end{bmatrix}$ Check equations:
$da + db + dc = d(a+b+c) = d \cdot 0 = 0$ ✓
$2(da) + 3(db) = d(2a+3b) = d \cdot 0 = 0$ ✓

(d) $\left\{ \begin{bmatrix} a-2b+c \\ b-2a-3c \\ a+b+c \end{bmatrix} : 2a + 3b + 4c = 0 \right\} = W$

✓(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ If $a = b = c = 0$, the equation is satisfied and the vector is all zero.

✓(+) Choose from W:
$\vec{u} = \begin{bmatrix} d-2e+f \\ e-2d-3f \\ d+e+f \end{bmatrix}$ $\vec{v} = \begin{bmatrix} g-2h+i \\ h-2g-3i \\ g+h+i \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} d-2e+f+g-2h+i \\ e-2d-3f+h-2g-3i \\ d+e+f+g+h+i \end{bmatrix} = \begin{bmatrix} (d+g)-2(e+h)+(f+i) \\ (e+h)-2(d+g)-3(f+i) \\ (d+g)+(e+h)+(f+i) \end{bmatrix}$
which is of the right form
$(a = d+g \quad b = e+h \quad c = f+i)$

Check equation:
$2(d+g) + 3(e+h) + 4(f+i)$
$= (2d+3e+4f) + (2g+3h+4i) = 0 + 0 = 0$ ✓

✓($\cdot$) Choose from W:
$\vec{u} = \begin{bmatrix} d-2e+f \\ e-2d-3f \\ d+e+f \end{bmatrix}$

$g \cdot \vec{u} = \begin{bmatrix} gd-2ge+gf \\ ge-2gd-3gf \\ gd+ge+gf \end{bmatrix}$ which is of the right form: $(a = gd \quad b = ge \quad c = gf)$

Check equation:
$2gd + 3ge + 4gf$
$= g \cdot (2d+3e+4f) = g \cdot 0 = 0$ ✓

2. $\vec{\mathbf{v}}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix}1\\2\\2\\1\end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$, $\vec{\mathbf{v}}_4 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $A = \begin{array}{c} \begin{matrix} A_1 & A_2 & A_3 \\ \shortparallel & \shortparallel & \shortparallel \end{matrix} \\ \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{bmatrix} \end{array}$.

(a) Which of these vectors are in Nul(A)? (Challenge: give the general answer)

$\vec{v}_1$ and $\vec{v}_2$ are the wrong size.

$$A\vec{v}_3 = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \checkmark$$

So of the vectors given, only v_3 is in Nul(A)

$$A\vec{v}_4 = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\15\\24\\33\end{bmatrix} \neq 0$$

(b) Which of these vectors are in Col(A)? (Challenge: give the general answer)

$\vec{v}_3$ and $\vec{v}_4$ are the wrong size.

The columns of A (and all vectors in their span) are of the form

$$V = a\begin{bmatrix}1\\4\\7\\10\end{bmatrix} + b\begin{bmatrix}2\\5\\8\\11\end{bmatrix}.$$

For v to be v_1 or v_2 we need

$(-4)(a + 2b = 1)$

$4a + 5b = 2$

$-3b = -2$

$b = 2/3$

$a + 2(2/3) = 1$

$a + 4/3 = 1 \quad a = -1/3$

$$V = -\frac{1}{3}\begin{bmatrix}1\\4\\7\\10\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\5\\8\\11\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = V_1$$

So of the vectors given, only V_1 is in Col (A)

(c) Give lots of examples (5 to infinitely many) of vectors in Nul(A):

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-4\\2\end{bmatrix}, \begin{bmatrix}-2\\4\\-2\end{bmatrix}, \ldots$$

Any multiple of v_3 is in there.

(d) Give lots of examples (5 to infinitely many) of vectors in Col(A):

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\4\\7\\10\end{bmatrix}, \begin{bmatrix}2\\5\\8\\11\end{bmatrix}, \begin{bmatrix}3\\6\\9\\12\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \ldots$$

All vectors of the form

$$a\begin{bmatrix}1\\4\\7\\10\end{bmatrix} + b\begin{bmatrix}2\\5\\8\\11\end{bmatrix}$$

3. Convert to nullspaces. Find a matrix B so that $\mathrm{Nul}(B)$ is as required:

(a) $\mathrm{Nul}(B) = \mathrm{Col}(A)$ where $A = \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \atop \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ -4 & -5 & -6 \end{bmatrix}$.

$A_1 + A_3 = 2A_2$
So the columns are linearly dependent

$$\mathrm{Col}(A) = \left\{ x\begin{bmatrix}1\\1\\4\\-4\end{bmatrix} + y\begin{bmatrix}2\\2\\5\\-5\end{bmatrix} : x,y \in \mathbb{R} \right\}$$

Looking for 2×4 B where $BA_1 = 0$, $BA_2 = 0$

$$\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 2\\ 4 & 5\\ -4 & -5\end{bmatrix} = \begin{bmatrix}0 & 0\\0&0\\0&0\\0&0\end{bmatrix} \qquad B = \begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

(b) $\mathrm{Nul}(B) = \left\{ \begin{bmatrix} a+b \\ c-d \\ a+c \\ b-d \end{bmatrix} : a+b+c+d = 0 \right\}$

$d = -a-b-c$

$$= \left\{ \begin{bmatrix} a+b \\ c-(-a-b-c) \\ a+c \\ b-(-a-b-c) \end{bmatrix} \text{ all } a,b,c \right\}$$

$$= \left\{ \begin{bmatrix} a+b \\ a+b+2c \\ a+c \\ a+2b+c \end{bmatrix} \text{ all } a,b,c \right\} = \left\{ a\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\\2\end{bmatrix} + c\begin{bmatrix}0\\2\\1\\1\end{bmatrix} \text{ all } a,b,c \right\}$$

So we need B where

$$\begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\\ 1 & 0 & 1\\ 1 & 2 & 1\end{bmatrix} =$$

$a + b + c + d = 0$
$a + b + 2d = 0$
$2b + c + d = 0$ $\Rightarrow$ $R1 = R2 \Rightarrow c = d$
$R1 = R3 \Rightarrow a = b$
$R2 = R3 \Rightarrow a = -c$

So for example $B = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$

(c) $\mathrm{Nul}(B) = \left\{ \begin{bmatrix} x \\ y \\ 3x+7y \\ 8x-9y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$$= \left\{ x\begin{bmatrix}1\\0\\3\\8\end{bmatrix} + y\begin{bmatrix}0\\1\\7\\9\end{bmatrix} : x,y \in \mathbb{R} \right\}$$

Need 2×4 B which sends both vectors to zero

$$\begin{bmatrix}0 & 1 & a & b\\ 1 & 0 & c & d\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 3 & 7\\ 8 & 9\end{bmatrix} \Rightarrow$$

Maybe this will work?

$3a + 8b = 0$
$7a + 9b = 1$ $\Rightarrow$

$3c + 8d = 1$
$7c + 9d = 0$ $\Rightarrow$

Inverse of $\begin{bmatrix}3 & 8\\ 7 & 9\end{bmatrix} = \frac{1}{-29}\begin{bmatrix}9 & -8\\ -7 & 3\end{bmatrix}$

So scale everything to make nicer

$$B = \begin{bmatrix}0 & 29 & -8 & 3\\ 29 & 0 & 9 & -7\end{bmatrix}$$

4. Convert to column spaces. Find a matrix B so that $\text{Col}(B)$ is as required:

(a) $\text{Col}(B) = \text{Nul}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$

RREF $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -5 & -10 & -15 & -20 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$

$x_1 = x_3 + 2x_4 + 3x_5$
$x_2 = -2x_3 - 3x_4 - 4x_5$
x_3, x_4, x_5 free

$$\text{Nul}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5 : \text{any } x_i \right\}$$

which is the column space of $B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\text{Col}(B) = \left\{ \begin{bmatrix} a+b \\ c-d \\ a+c \\ b-d \end{bmatrix} : a+b+c+d = 0 \right\}$

$d = -a-b-c$, so

$$\text{Col}(B) = \left\{ \begin{bmatrix} a+b \\ c-(-a-b-c) \\ a+c \\ b-(-a-b-c) \end{bmatrix} \text{ all } a,b,c \right\}$$

$$= \left\{ \begin{bmatrix} a+b \\ a+b+2c \\ a+c \\ a+2b+c \end{bmatrix} \text{ all } a,b,c \right\}$$

$$= \left\{ a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \text{ all } a,b,c \right\}$$

This is the column space of

$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(c) $\text{Col}(B) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : c = 3a+5b, d = 7a+9b \right\}$

$$= \left\{ \begin{bmatrix} a \\ b \\ 3a+5b \\ 7a+9b \end{bmatrix} \text{ all } a,b \right\} = \left\{ a\begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 5 \\ 9 \end{bmatrix} \text{ all } a,b \right\}$$

This is the column space of $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 5 \\ 7 & 9 \end{bmatrix}$