An **inner product** is an operation that takes two vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ from the same vector space V and returns a number $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$.

It should satisfy the following properties for vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ in V and a scalar number c:

- $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
- $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{w}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$
- $(c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$

It follows from these properties that for any vector $\vec{\mathbf{u}}$, the transformation $\vec{\mathbf{u}}^T : V \to \mathbb{R} : \vec{\mathbf{v}} \mapsto \vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ is a linear transformation, so $\vec{\mathbf{u}}^T$ must be a matrix.

If $V = \mathbb{R}^n$, then the **standard inner product** takes $\vec{\mathbf{u}}^T$ to be the $1 \times n$ matrix whose entries are the same as $\vec{\mathbf{u}}$, just as a row instead of a column. So if $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}}^T \vec{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

If V consists of functions f so that $|f|^2$ is integrable, then its standard inner product is actually the integral: $\vec{\mathbf{f}} \cdot \vec{\mathbf{g}} = \int f(x) g(x) \, \mathrm{d}x$. Electrical will use this a lot (Fourier series).

If V is space-time coordinates, then $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}^T = \begin{bmatrix} x & y & z & -t \end{bmatrix}$ so that

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 - t_1 t_2$$

We'll be sticking with the standard inner product which has the following two important features (symmetric, definite):

- $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
- $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} > 0$ unless $\vec{\mathbf{u}} = \vec{\mathbf{0}}$

For such a nice inner product, we can also define the **length** of a vector as $\|\vec{\mathbf{v}}\| = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$, the distance between two vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ as the length of $\vec{\mathbf{u}} - \vec{\mathbf{v}}$, and the angle between them as the angle θ so that $\cos(\theta) = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\|}$.

In general, we say that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal** if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$. This basically means that $\theta = \pm 90°$ (only exception being when $\|\vec{\mathbf{u}}\|$ or $\|\vec{\mathbf{v}}\|$ is 0).

A **unit length** vector is a vector $\vec{\mathbf{v}}$ of length $1 = \|\vec{\mathbf{v}}\|$.

HW6.1 # 1,3,5,7,9,11

The most interesting things happen when we have a bunch of unit length vectors that are all orthogonal to each other. For example, let's look at this basis of $\mathbb{R}^4$:

$$\vec{\mathbf{a}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{\boldsymbol{\ell}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{\mathbf{m}} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{h}} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Can you write $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ as a linear combination of $\vec{\mathbf{a}}$, $\vec{\boldsymbol{\ell}}$, $\vec{\mathbf{m}}$, $\vec{\mathbf{h}}$?

What if you weren't allowed to use $\vec{\mathbf{h}}$? What is the closest you could get to being a linear combination of $\vec{\mathbf{a}}$, $\vec{\boldsymbol{\ell}}$, $\vec{\mathbf{m}}$?

What if you were only allowed to use $\vec{\mathbf{a}}$? What is the closest you could get to being a linear combination of $\vec{\mathbf{a}}$?

These are called the **projections** onto the span of the vectors you are allowed to use, and they can be computed as in theorem 5, page 339 in the book.

HW 6.2 # 1,3,5,7,9

MA322-007 Apr 7 Quiz Name:________________________

6.1a Find a vector that is orthogonal to $\vec{\mathbf{v}} = (3, 4)$. Can you find one that is unit length?

6.2a Find a vector that is orthogonal to $\vec{\mathbf{u}} = (4, 4, 7)$ and $\vec{\mathbf{w}} = (8, -1, -4)$. Can you find one that is unit length?

Hint: $\vec{\mathbf{x}} = (1, 1, 1)$ is wrong, since it points in both the $\vec{\mathbf{u}}$ and the $\vec{\mathbf{w}}$ directions. Could remove the wrongness?