MA322-001 Apr 11 Quiz   Name: ______________________

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$. Want to solve $\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 4 & 3 & 11 \\ 0 & 0 & 13 \end{array}\right]$

1. Find vectors $\vec{g}_1$ and $\vec{g}_2$ with the same span as $\vec{v}_1$ and $\vec{v}_2$, except with $\vec{g}_1 \cdot \vec{g}_2 = 0$.

$$\vec{g}_1 = \vec{v}_1$$
$$\vec{g}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{g}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - \frac{15}{25}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - 9/5 \\ 3 - 12/5 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$$

$$\vec{g}_1 \cdot \vec{g}_1 = 25$$
$$\vec{g}_1 \cdot \vec{g}_2 = 0 \qquad \vec{g}_2 \cdot \vec{g}_2 = 1$$

Orthogonal, but not Orthonormal

2. What is the projection of $\vec{b}$ onto $\vec{g}_1$? "in the direction of $\vec{g}_1$"

$$\mathrm{proj}_{\vec{g}_1}(\vec{b}) = \frac{\vec{b} \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 = \frac{21 + 44}{25}\vec{g}_1 = \boxed{\frac{13}{5}\vec{g}_1} = \begin{bmatrix} 39/5 \\ 52/5 \\ 0 \end{bmatrix}$$

3. What is the projection of $\vec{b}$ onto $\vec{g}_2$?

$$\mathrm{proj}_{\vec{g}_2}(\vec{b}) = \frac{\vec{b} \cdot \vec{g}_2}{\vec{g}_2 \cdot \vec{g}_2}\vec{g}_2 = \frac{-\frac{28}{5} + \frac{33}{5}}{1}\vec{g}_2 = \vec{g}_2 = \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$$

4. Define $\vec{g}_3 = \vec{b} - \mathrm{proj}_{\vec{g}_1}(\vec{b}) - \mathrm{proj}_{\vec{g}_2}(\vec{b})$

$$\vec{g}_3 = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix} - \begin{bmatrix} 39/5 \\ 52/5 \\ 0 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$$

Notice $\vec{g}_1 \cdot \vec{g}_3 = 0$, $\vec{g}_2 \cdot \vec{g}_3 = 0$ so $\begin{bmatrix} | & | & | \\ \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ | & | & | \end{bmatrix}$ has orthogonal columns (but not orthonormal)

5. If you write $\vec{\mathbf{b}} = y_1\vec{\mathbf{g}}_1 + y_2\vec{\mathbf{g}}_2 + y_3\vec{\mathbf{g}}_3$, what are y_1, y_2, and y_3?

$\text{proj}_{\vec{g}_1}(\vec{b})$ $\quad \text{proj}_{\vec{g}_2}(\vec{b})$ $\quad$ etc

$$\vec{b} = \frac{13}{5}\vec{g}_1 + 1\vec{g}_2 + 1\vec{g}_3$$

6. Write each $\vec{\mathbf{v}}_i$ in terms of the $\vec{\mathbf{g}}_j$s

$$\vec{g}_1 = \vec{V}_1$$
$$\vec{g}_2 = \vec{V}_2 - \frac{15}{25}\vec{g}_1$$
$$\vec{g}_3 = \vec{V} - \frac{13}{5}g_1 - \vec{g}_2$$

$$\vec{V}_1 = \vec{g}_1$$
$$\vec{V}_2 = \frac{15}{25}\vec{g}_1 + \vec{g}_2$$
$$\vec{V} = \frac{13}{5}\vec{g}_1 + \vec{g}_2 + \vec{g}_3$$

7. Find the best x_1, x_2 so that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$ is as close to $\vec{\mathbf{b}}$ as possible.

$$X_1\vec{V}_1 + X_2\vec{V}_2$$
$$= X_1(\vec{g}_1) + X_2\left(\frac{15}{25}\vec{g}_1 + \vec{g}_2\right)$$
$$= \left(X_1 + \frac{15}{25}X_2\right)\vec{g}_1 + (X_2)\vec{g}_2$$

so $X_1 + \frac{15}{25}X_2 = \frac{13}{5}$

$X_2 = 1$

So $X_1 = 2$, $X_2 = 1$

8. How far from $\vec{\mathbf{b}}$ must it be?

$$\vec{b} \approx 2\begin{bmatrix}3\\4\\0\end{bmatrix} + \begin{bmatrix}1\\3\\0\end{bmatrix} = \begin{bmatrix}7\\11\\0\end{bmatrix}$$

$$\left\|\vec{b} - \left(\frac{13}{5}\vec{g}_1 + \vec{g}_2\right)\right\| = \sqrt{\left(\frac{13}{5} - \frac{13}{5}\right)^2\vec{g}_1\cdot\vec{g}_1 + (1-1)^2\vec{g}_2\cdot\vec{g}_2 + (1-0)\vec{g}_3\cdot\vec{g}_3}$$

$$= \|\vec{g}_3\| = 13$$

The error is $\vec{g}_3 = \begin{bmatrix}0\\0\\13\end{bmatrix}$