What are the eigenvalues and eigenvectors of the matrix $\vec{g}_1 \cdot \vec{g}_1^T$?

$(\lambda = \|g_1\|^2, \vec{g}_1)$

$(\lambda = 0$, all perpendicular $\vec{g}_2, \ldots, \vec{g}_n)$

$(\vec{g}_1 \cdot \vec{g}_1^T)\vec{v} = \vec{g}_1(\vec{g}_1^T\vec{v})$

$= (\vec{g}_1 \cdot \vec{v})\vec{g}_1$

is a multiple of $\vec{g}_1$

so all eigenvectors are either

$\vec{g}_1$ with $\lambda = \vec{g}_1 \cdot \vec{g}_1$

or $\vec{g}_2$ with $\lambda = \vec{g}_1 \cdot \vec{g}_2 = 0$

If $\vec{g}_i$ form an orthonormal system and λ_i are numbers, what are the eigenvalues and eigenvectors of $\sum \vec{g}_i \lambda_i \vec{g}_i^T$?

$(\lambda_i, \vec{g}_i)$

$$\left(\sum \vec{g}_i \lambda_i \vec{g}_i^T\right)\vec{g}_j = \sum \left(\lambda_i(\vec{g}_i \cdot \vec{g}_j)\right)\vec{g}_i$$
$$= \lambda_j(\vec{g}_j \cdot \vec{g}_j)\vec{g}_j$$
$$= \lambda_j (1) \vec{g}_j$$

If $G = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ \vec{g}_1 & \cdots & \vec{g}_n \\ \downarrow & \cdots & \downarrow \end{bmatrix}$ and D is the diagonal matrix with entries λ_i, then what are the eigenvalues and eigenvectors of GDG^T? Same matrix!

$$GDG^T \begin{bmatrix} \uparrow \\ \vec{g}_i \\ \downarrow \end{bmatrix} = GD \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}_{i\text{th}} = G \begin{bmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \end{bmatrix} = \lambda_i \vec{g}_i$$

An important theorem is that **every symmetric matrix** can be factored as GDG^T. For applications, it is nice to know that this can be done efficiently and accurately even for extremely large matrices. In this chapter, we'll assume that is already done, and see what we get for free.

A standard example of a symmetric matrix A is an **energy functional** that takes a vector $\vec{\mathbf{v}}$ to its energy $A(\vec{\mathbf{v}}) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. If $A = I_n$ is the identity, then the energy just becomes $\|\vec{\mathbf{v}}\|^2$, the length. Many applied problems have two different energy measurements (I think of them as cost and revenue, how much you put in versus how much you get out).

How do you find the vector $\vec{\mathbf{v}}$ such $\vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ is maximized, given a budget $\vec{\mathbf{v}}^T B \vec{\mathbf{v}} = b$?

Step 1: Find β such that $B = \beta^T \beta$. Set $\vec{\mathbf{w}} = \beta \vec{\mathbf{v}}$, so $\vec{\mathbf{v}} = \beta^{-1}\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}^T B \vec{\mathbf{v}} = \vec{\mathbf{v}}^T \beta^T \beta \vec{\mathbf{v}} = \|\vec{\mathbf{w}}\|^2$

Step 2: Find G, D such that $\beta^{-T} A \beta^{-1} = G D G^T$. Then $\vec{\mathbf{v}}^T A \vec{\mathbf{v}} = \vec{\mathbf{w}}^T \beta^{-T} A \beta^{-1} \vec{\mathbf{w}} = \vec{\mathbf{w}} G D G^T \vec{\mathbf{w}} = \sum_i \lambda_i (\vec{\mathbf{g}}_i \cdot \vec{\mathbf{w}})$.

The simple answer is that we find the biggest λ_i, and then set $\vec{\mathbf{w}} = \sqrt{b}\,\vec{\mathbf{g}}_i$. Any other $\vec{\mathbf{g}}_j$ are just wasting length. That means $\vec{\mathbf{v}} = \sqrt{b}\,\beta^{-1}\vec{\mathbf{g}}_i$.

For $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 14 \end{bmatrix}$, $b = 100$, and $A = \begin{bmatrix} 3.0224 & 1.6592 & -4.8880 \\ 1.6592 & 4.1136 & 8.2960 \\ -4.8880 & 8.2960 & 58.5600 \end{bmatrix}$, find $\vec{\mathbf{v}}$ so that $\vec{\mathbf{v}}^T B \vec{\mathbf{v}} \le b$ and $\vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ is maximized.

Check That $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}}_{\beta^T} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 14 \end{bmatrix}}_{B}$

Step 1: $\beta = \text{chol}(B)$, $\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Step 2: $[g, d] = \text{eig}(\beta' \backslash A / \beta)$ $\quad g = \begin{array}{c} \begin{matrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{matrix} \\ \begin{bmatrix} -0.48 & 0.80 & 0.36 \\ 0.64 & 0.60 & -0.48 \\ 0.60 & 0.00 & 0.80 \end{bmatrix} \end{array}$

$d = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $\left(\lambda = 7, \begin{bmatrix} -0.48 \\ 0.64 \\ 0.60 \end{bmatrix}\right)$ is best eigenpair

$\vec{w} = \begin{bmatrix} -0.48 \\ 0.64 \\ 0.60 \end{bmatrix} \cdot 10$ has $\|\vec{w}\|^2 = 100$

so $\vec{v} = \beta^{-1}\vec{w} = \begin{bmatrix} -9.2 \\ 2.4 \\ 2.0 \end{bmatrix}$ has $v^T B v = 100$

and $v^T A v = 700 = \lambda (v^T B v)$

Check your work: the best $\vec{\mathbf{v}}$ has $\vec{\mathbf{v}}^T A \vec{\mathbf{v}} = 700$.