Matrix Operations

Recall that an $m \times n$ ***matrix*** $\boldsymbol{A}$ is a rectangular array of mn scalars arranged in m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

We will also denote the $m \times n$ ***matrix*** $\boldsymbol{A}$ by

$$A = \left(a_{ij} \right).$$

In the case $m = n$ we call $A = \left(a_{ij} \right)$ a ***square matrix***. If all the entries of the $m \times n$ ***matrix*** $\boldsymbol{A}$ are zero, we call $\boldsymbol{A}$ the ***zero matrix*** and denote it by $\mathbf{0}_{\boldsymbol{m} \times \boldsymbol{n}}$. The term *matrix* was first used in 1850 by *James Joseph Sylvester* to differentiate matrices from determinants.

Example

The matrix $\boldsymbol{A}$ shown below is of size 3×2.

$$A = \begin{pmatrix} -2 & 3 \\ 5 & -1 \\ 2 & -3 \end{pmatrix}$$

We observe, for example, that $a_{12} = 3$, $a_{22} = -1$, and $a_{31} = 2$.

Definition

Suppose that $A = \left(a_{ij} \right)$ and $B = \left(b_{ij} \right)$ are both $m \times n$ matrices and $\alpha \in \mathbb{R}$.

1. We say $A = B$ if and only if $a_{ij} = b_{ij} \quad (1 \le i \le m,\ 1 \le j \le n)$.

2. The ***matrix sum*** of ***A*** and ***B***, denoted by $A + B$, is the $m \times n$ matrix given by

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1j}+b_{1j} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2j}+b_{2j} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \dots & a_{ij}+b_{ij} & \dots & a_{in}+b_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mj}+b_{mj} & \dots & a_{mn}+b_{mn} \end{pmatrix} = \left(a_{ij} + b_{ij} \right).$$

3. The product of the scalar " with the matrix ***A*** is given by

$$\alpha\ \mathrm{A} = \begin{pmatrix} \alpha\, a_{11} & \alpha\, a_{12} & \dots & \alpha\, a_{1j} & \dots & \alpha\, a_{1n} \\ \alpha\, a_{21} & \alpha\, a_{22} & \dots & \alpha\, a_{2j} & \dots & \alpha\, a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha\, a_{i1} & \alpha\, a_{i2} & \dots & \alpha\, a_{ij} & \dots & \alpha\, a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha\, a_{m1} & \alpha\, a_{m2} & \dots & \alpha\, a_{mj} & \dots & \alpha\, a_{mn} \end{pmatrix} = \left(\alpha\ a_{ij} \right).$$

We note that the matrix sum is defined only when the two matrices are of the same size. In this case, matrix addition is performed by adding corresponding components of the matrices ***A*** and ***B***.

Examples

1. $$\begin{pmatrix} 5 & 1 & 6 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3+2 & 8-7 & 2(3) \\ 1+2 & 0-2 & 1 \end{pmatrix}$$

2. $$\begin{pmatrix} 2 & 8 \\ 6 & 9 \end{pmatrix} \neq \begin{pmatrix} 8 & 2 \\ 6 & 9 \end{pmatrix}$$

3. $$\mathbf{0}_{2\times 3} \neq \mathbf{0}_{6\times 6}$$

4. $$\begin{pmatrix} 5 & 1 & 6 \\ 3 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 6 & -5 & 4 \\ -5 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 11 & -4 & 10 \\ -2 & 1 & 8 \end{pmatrix}$$

5. $$\begin{pmatrix} 2 & 8 \\ 6 & 9 \end{pmatrix} + \mathbf{0}_{2\times 2} = \begin{pmatrix} 2 & 8 \\ 6 & 9 \end{pmatrix}$$

6. $$\begin{pmatrix} 4 & -5 & 8 \\ 2 & 6 & -7 \\ -1 & 0 & 2 \end{pmatrix} + 3\begin{pmatrix} -1 & 2 & 9 \\ 0 & 0 & 0 \\ -3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -5 & 8 \\ 2 & 6 & -7 \\ -1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -3 & 6 & 27 \\ 0 & 0 & 0 \\ -9 & 12 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 35 \\ 2 & 6 & -7 \\ -10 & 12 & 17 \end{pmatrix}$$

Theorem

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be $m \times n$ matrices and let $\alpha, \beta \in \mathbb{R}$. Then

1. $A + \mathbf{0}_{m \times n} = A$

2. $0\, A = \mathbf{0}_{m \times n}$

3. $A + B = B + A$

4. $(A + B) + C = A + (B + C)$

5. $\alpha\, (A + B) = \alpha\, A + \alpha\, B$

6. $(\alpha + \beta)\, A = \alpha\, A + \beta\, A$

6. $1\, A = A$

Proof of 3

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1j} + b_{1j} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2j} + b_{2j} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} + b_{i1} & a_{i2} + b_{i2} & \dots & a_{ij} + b_{ij} & \dots & a_{in} + b_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mj} + b_{mj} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} + a_{11} & b_{12} + a_{12} & \dots & b_{1j} + a_{1j} & \dots & b_{1n} + a_{1n} \\ b_{21} + a_{21} & b_{22} + a_{22} & \dots & b_{2j} + a_{2j} & \dots & b_{2n} + a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i1} + a_{i1} & b_{i2} + a_{i2} & \dots & b_{ij} + a_{ij} & \dots & b_{in} + a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} + a_{m1} & b_{m2} + a_{m2} & \dots & b_{mj} + a_{mj} & \dots & b_{mn} + a_{mn} \end{pmatrix} = B + A$$

The above is rather verbose. Alternatively, we see that

$$A + B = \left(a_{ij}\right) + \left(b_{ij}\right) = \left(a_{ij} + b_{ij}\right) = \left(b_{ij} + a_{ij}\right) = \left(b_{ij}\right) + \left(a_{ij}\right) = B + A.$$

Problem

1. Let $A = \begin{pmatrix} 3 & -5 & 8 \\ -2 & 7 & 9 \end{pmatrix}$. Verify that $A + A + A = 3\,A$.

2. True or False: For any matrix ***A*** and any natural number *n*, we have that

$$\sum_{k=1}^{n} A = n\,A.$$

Problem

Solve for ***X*** in the equation $3\,X + A = B$ where

$$A = \begin{pmatrix} 3 & 7 & 2 & -1 \\ -4 & 5 & 8 & -5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -7 & 4 & 2 \\ -1 & 2 & 2 & -3 \end{pmatrix}.$$

Definition

Let $A = \left(a_{ij}\right)$ be an $m \times n$ matrix and let $B = \left(b_{ij}\right)$ be an $n \times p$ matrix. Then the ***matrix product*** of ***A*** and ***B***, denoted by $A\,B$, is an $m \times p$ matrix $C = \left(c_{ij}\right)$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

First, we observe that the matrix product is only defined when the number of columns of the first matrix equals the number rows of the second matrix. Second, the we see that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \left(a_{i1}\ a_{i2}\ \dots\ a_{in}\right) \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}.$$

In other words, the ij^{th} component of the matrix product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the inner product of the i^{th} row of $\boldsymbol{A}$ with the j^{th} column of $\boldsymbol{B}$.

Examples

1. $$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1(-2)+2(2) & 1(4)+2(-2) \\ 1(-2)+1(2) & 1(4)+1(-2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. $$\begin{pmatrix} -2 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

3. $$\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 4 & -8 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1(4)+4(-1) & 1(-8)+4(2) \\ 2(4)+8(-1) & 2(-8)+8(2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. $$\begin{pmatrix} 4 & -8 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} -12 & -48 \\ 3 & 12 \end{pmatrix}$$

5. $$\begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{pmatrix} = \begin{pmatrix} 1(0)+0(3) & 1(6)+0(8) & 1(1)+0(-2) \\ -2(0)+3(3) & -2(6)+3(8) & -2(1)+3(-2) \\ 5(0)+4(3) & 5(6)+4(8) & 5(1)+4(-2) \\ 0(0)+1(3) & 0(6)+1(8) & 0(1)+1(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{pmatrix}$$

6. $$\begin{pmatrix} 1 & 5 & -3 \\ -4 & 2 & 8 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -3 \\ -4 & 2 & 8 \\ 1 & 0 & -3 \end{pmatrix}$$

Problem

Find a matrix ***A***, if possible, that commutes with the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$. How many matrices commute under matrix multiplication with $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$?

Example

A certain retail store sells brand X SLR 35mm cameras and DSLR digital cameras. The following matrix on the left gives the sales of these items for three months of Januray, February, March; the matrix on the right gives the sales price and the dealer cost of these items.

$$\begin{array}{cc} & \begin{array}{ccc} \text{J} & \text{F} & \text{M} \end{array} \\ \begin{array}{c} \textit{SLR 35mm} \\ \textit{DSLR} \end{array} & \begin{pmatrix} 5 & 12 & 10 \\ 9 & 17 & 16 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} \text{35mm} & \text{DSLR} \end{array} \\ \begin{array}{c} \textit{price} \\ \textit{cost} \end{array} & \begin{pmatrix} \$781 & \$1529 \\ \$567 & \$1376 \end{pmatrix} \end{array}$$

We form the product (in the only order that makes sense!):

$$\begin{pmatrix} 781 & 1529 \\ 567 & 1376 \end{pmatrix} \begin{pmatrix} 5 & 12 & 10 \\ 9 & 17 & 16 \end{pmatrix} = \begin{pmatrix} 17666 & 35365 & 32274 \\ 15219 & 30196 & 27686 \end{pmatrix}$$

So what?!!? Well, we see that the product yields

$$\begin{array}{cc} & \begin{array}{ccc} \text{J} & \text{F} & \text{M} \end{array} \\ \begin{array}{c} \textit{total sales} \\ \textit{dealer cost} \end{array} & \begin{pmatrix} \$17{,}666 & \$35{,}365 & \$32{,}274 \\ \$15{,}219 & \$30{,}196 & \$27{,}686 \end{pmatrix} \end{array}$$

Problem

Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$. Show that $AC = BC$.

Conclusion: $AC = BC$ yet $A \neq B$. Hence, in general, cancellation is not valid for matrix products.

Theorem

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be matrices (with sizes so that the given products below are well-defined) and let $\alpha \in \mathbb{R}$. Then

1. $A\,(B\,C) = (A\,B)\,C$
2. $A\,(B + C) = A\,B + A\,C$
3. $(A + B)\,C = A\,C + B\,C$
4. $\alpha\,(A\,B) = (\alpha\,A)\,B = A\,(\alpha\,B)$

Definition

Let $\boldsymbol{A}$ be any square matrix and let k be any positive integer. Then we define A^k as

$$A\,A\,A\,\ldots\,A \quad (k \text{ times}).$$

(The above is well-defined.)

Example

1.
$$\begin{pmatrix} 1 & -5 \\ -3 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & -5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ -3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 16 & -15 \\ -9 & 19 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ -3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 61 & -110 \\ -66 & 83 \end{pmatrix}$$

2. Let $A = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 & 0 & 27 & 0 \\ 0 & 0 & 0 & 0 & 27 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$A^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 81 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, and $A^k = \mathbf{0}_{5 \times 5}$ $(k \geq 5)$. The matrix $\boldsymbol{A}$ is an example of a ***nilpotent***

matrix.

Problem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two 3×3 matrices.

1. True or False: $(A - B)(A + B) = A^2 - B^2$

2. True or False: $(A + B)^2 = A^2 + 2AB + B^2$

Theorem

Suppose that $\boldsymbol{A}$ is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix and $\boldsymbol{I_k}$ is a $k \times k$ matrix with $I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Then

1. $A I_n = A$

2. $I_m A = A$.