Transpose

Since it is sometimes convenient to flip a matrix over its main diagonal, transforming rows into columns and visa versa (the i^{th} row becoming the i^{th} column and the j^{th} column becoming the j^{th} row), we make the following

Definition.

The ***transpose*** of the m × n matrix $A = (a_{ij})$ is the n × m matrix given by $A^T = C = (c_{ij})$ where $c_{ij} = a_{ji}$.

Examples

1. $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$ Note, for example, the entry 6 in the 2,3 position of the original matrix becomes the entry in the 3,2 position of the transpose.

2. $$\begin{pmatrix} 13 & 64 & 73 \\ -5 & 28 & -9 \end{pmatrix}^T = \begin{pmatrix} 13 & -5 \\ 64 & 28 \\ 73 & -9 \end{pmatrix}$$

3. $$\begin{pmatrix} 22 \\ -45 \\ 17 \\ 98 \end{pmatrix}^T = \begin{pmatrix} 22 & -45 & 17 & 98 \end{pmatrix}$$

Properties of the Transpose

In each of the following, suppose that A and B are matrices whose sizes make the operation well-defined.

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $(AB)^T = B^TA^T$
4. $(A^{-1})^T = (A^T)^{-1}$

Proof

We assume the standard notation setting $A = (a_{ij})$ and $B = (b_{ij})$.

1. Let $A + B = C$ where $C =(c_{ij})$ so that we have $c_{ij} = a_{ij} + b_{ij}$. Then $(A + B)^T = C^T = (c_{ji}) = (a_{ji} + b_{ji}) = (a_{ji}) + (b_{ji}) = (a_{ij})^T + (b_{ij})^T = A^T + B^T$.
2. Set $A^T = C = (c_{ij})$ with $c_{ij} = a_{ji}$. We get $(A^T)^T = C^T = (c_{ij})^T = (a_{ji})^T = (a_{ij}) = A$.
3. We use $\mathbf{r}_{\mathbf{Ai}}$ to denote the i^{th} row of A and $\mathbf{c}_{\mathbf{Aj}}$ for the j^{th} column of A and employ the same notation for the rows and columns of B, simply replacing the A in the subscript with a B. Then the i, j^{th} entry in the product AB is the dot product $\mathbf{r}_{\mathbf{Ai}} \cdot \mathbf{c}_{\mathbf{Bj}}$ and so is the j, i^{th} entry of $(AB)^{T.}$ Meanwhile, the corresponding entry in the product B^TA^T is the dot product of the j^{th} row of B^T which is the j^{th} column of B, namely, $\mathbf{c}_{\mathbf{Bj}}$**,** with the i^{th} column of A^T which is the i^{th} row of A, $\mathbf{r}_{\mathbf{Ai}}$. But since the dot product is commutative, $\mathbf{r}_{\mathbf{Ai}} \cdot \mathbf{c}_{\mathbf{Bj}} = \mathbf{c}_{\mathbf{Bj}} \cdot \mathbf{r}_{\mathbf{Ai}}$, and we see that the entries of $(AB)^T$ and B^TA^T are equal; thus, $(AB)^T = B^TA^T$.
4. Finally, observe that, using the result from part 3, $(AA^{-1})^T = (A^{-1})^TA^T$. But $AA^{-1} = I$ and, clearly $I^T = I$, so $(A^{-1})^TA^T = I$. Consequently, $(A^{-1})^T$ acts like the inverse of A^T and so, since inverses are unique, $(A^T)^{-1} = (A^{-1})^{T.}$

Definition

The matrix A is said to be ***symmetric*** provided $A^T = A$.

From the definition it is clear that for a matrix to be symmetric, it must be square. We conclude this section by pointing out one way of creating symmetric matrices. Given any matrix, A, the product A^TA is symmetric since $(A^TA)^T = A^T(A^T)^T = A^T A$.

Example

$$\begin{pmatrix} 3 & -7 \\ 5 & 10 \end{pmatrix}^T \begin{pmatrix} 3 & -7 \\ 5 & 10 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -7 & 10 \end{pmatrix} \begin{pmatrix} 3 & -7 \\ 5 & 10 \end{pmatrix} = \begin{pmatrix} 34 & 29 \\ 29 & 59 \end{pmatrix}$$