Intro to Contemporary Math

Fair and Envy-Free Arrangements (Review)

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Agenda

- Creating Fair Arrangements
  - Picking Winners using the Average Bid
  - Paying the Losers their Fair Shares
  - The Winner's Compensation as a Leftover
- Creating Envy-Free Arrangements
  - Picking Winners using the Highest Bid
  - Paying the Losers using Top Two Fair Shares
  - The Winner's Compensation as a Leftover

Announcements

- Homework due tonight (Wednesday)
- Exam on Friday (bring a calculator)

Fair Arrangement (Intro)

Two people, Alice and Bob, are fighting over an item. They make the following bids:

Name	Bids	Fair Share	Compensation
Alice	12		
Bob	50		

They want a Fair arrangement.

Fair Arrangement (Fair Shares)

Name	Bids	Fair Share	Compensation
Alice	12	6	
Bob	50	25	

Let's compute their fair shares. This is their bid divided by the number of people:

$$\textit{Alice's fair share} = \frac{12,\ \textit{her bid}}{2\ \textit{people}} = 6$$

$$\textit{Bob's fair share} = \frac{50,\ \textit{his bid}}{2\ \textit{people}} = 25$$

Fair Arrangement (Fair Shares as a Goal)

Name	Bids	Fair Share	Compensation
Alice	12	6	
Bob	50	25	

Alice wants a minimum of 6 dollars worth from items and cash, and Bob wants a minimum of 25 dollars worth from items and cash. For the arrangement to be Fair, their compensations must be greater than or equal to their fair shares.

Fair Arrangement (Choose Winner)

Name	Bids	Fair Share	Compensation
Alice	12	6	
Bob	50	25	

In a Fair arrangement, the winner can be anyone who bid more than the average bid. Let's compute that:

$$m = \textit{Average Bid} = \frac{12+50}{2} = \frac{62}{2} = 31.$$

Since Bob's bid of 50 is larger than the average bid of 31, Bob can be chosen as the winner.
In general, if there are more people involved, their could be more than one choice for the possible winner.

Fair Arrangement (Paying Others)

Name	Bids	Fair Share	Compensation
Alice	12	6	6 from Bob
Bob	50	25	

Bob has the item. Alice is empty-handed. Bob must pay Alice.

- Alice needs 6, her fair share. Since she has no item, she will depend on the winner, Bob, to pay her enough.
- Thus, Bob must pay Alice 6. If there are more people, Bob would pay each other person their fair shares.
- Alice's compensation is what Bob pays her, since she has no item:

$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{6}_{From\ Bob} = 6\ from\ Bob$$

Fair Arrangement (Winner's Compensation)

Name	Bids	Fair Share	Compensation
Alice	12	6	6 from Bob
Bob	50	25	44 left

Bob's compensation might not be his fair share. He is not paying himself!

- Bob has the item, which is worth 50 to him (his winning bid).
- Bob lost 6 when paying Alice.
- Thus, Bob's compensation is what he has left after paying the others:

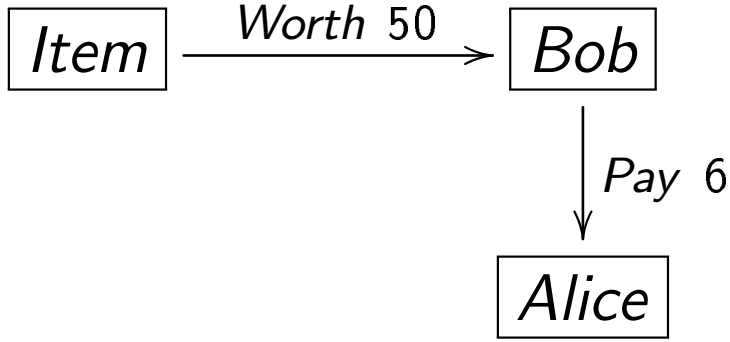
$$x_{Bob} = \underbrace{50}_{Item} - \underbrace{6}_{To\ Alice} = 44\ left$$

Fair Arrangement (Conclusion)

Name	Bids	Fair Share	Compensation
Alice	12	6	6 from Bob
Bob	50	25	44 left

Notice that Alice and Bob's compensations are greater than or equal to their fair shares. This arrangement is Fair.

Fair Arrangement Diagram



$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{6}_{From\ Bob} = 6\ from\ Bob$$

$$x_{Bob} = \underbrace{50}_{Item} - \underbrace{6}_{To\ Alice} = 44\ left$$

Envy-Free Arrangement (Intro)

Three people, Alice, Bob, and Carla, are fighting over an item. They make the following bids:

Name	Bids	Fair Share	Compensation
Alice	12		
Bob	21		
Carla	24		

They want an Envy-free arrangement.

Envy-Free Arrangement (Fair Shares)

Name	Bids	Fair Share	Compensation
Alice	12	4	
Bob	21	7	
Carla	24	8	

Let us compute the fair shares. This is their bid divided by the number of people:

$$Alice's\ fair\ share = \frac{12,\ her\ bid}{3\ people} = 4$$

$$Bob's\ fair\ share = \frac{21,\ his\ bid}{3\ people} = 7$$

$$Carla's\ fair\ share = \frac{24,\ her\ bid}{3\ people} = 8$$

We will need some (but not all) of these fair shares later.

Envy-Free Arrangement (Choose Winner)

Name	Bids	Fair Share	Compensation
Alice	12	4	
Bob	21	7	
Carla	24	8	

In an Envy-free arrangement, only the highest bidder can win. Thus, we have to choose Carla as the winner.

Envy-Free Arrangement (Paying Others)

Name	Bids	Fair Share	Compensation
Alice	12	4	
Bob	21	7	
Carla	24	8	

Carla must pay the others:

- the same dollar amount,
- between the top two fair shares

Thus, Carla should pay Alice and Bob between 7 and 8 dollars.

Envy-Free Arrangement (Paying Others)

Name	Bids	Fair Share	Compensation
Alice	12	4	7 from Carla
Bob	21	7	7 from Carla
Carla	24	8	

Thus, Carla should pay Alice and Bob between 7 and 8 dollars. For this example, let us have her pay 7 each to Alice and Bob. Their compensations will be what Carla pays them, since they have no item:

$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

$$x_{Bob} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

Envy-Free Arrangement (Winner's Compensation)

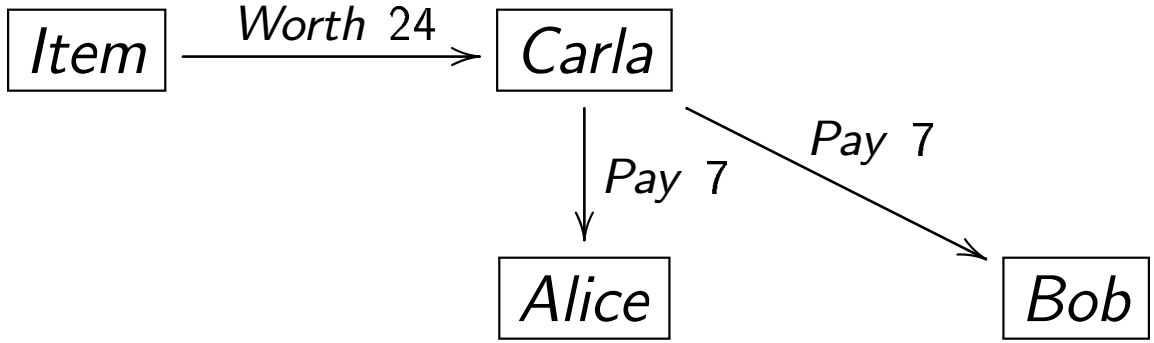
Name	Bids	Fair Share	Compensation
Alice	12	4	7 from Carla
Bob	21	7	7 from Carla
Carla	24	8	10 left

Carla is not going to pay herself 7 dollars!

- Carla has the item, which is worth 24 to her.
- Carla lost 7 when paying Alice, and lost 7 when paying Bob.
- Thus, Carla's compensation is what she has left after paying the others:

$$x_{Carla} = \underbrace{24}_{Item} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 10\ left$$

Envy-free Arrangement Diagram



$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

$$x_{Bob} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

$$x_{Carla} = \underbrace{24}_{Item} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 10\ left$$

No Envy (Carla)

Name	Bids	Fair Share	Compensation
Alice	12	4	7 from Carla
Bob	21	7	7 from Carla
Carla	24	8	10 left

Carla's (actual) compensation is

$$x_{Carla} = \underbrace{24}_{Item} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 10\ left$$

Alice uses her own bid when trying to compute Carla's compensation. Alice thinks Carla got:

$$\underbrace{12}_{Alice's\ bid} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = -2\ left$$

Bob uses his own bid when trying to compute Carla's compensation. Bob thinks Carla got:

$$\underbrace{21}_{Alice's\ bid} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 7\ left$$

Carla's (actual) compensation is

$$x_{Carla} = \underbrace{24}_{Item} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 10\ left$$

Alice uses her own bid when trying to compute Carla's compensation. Alice thinks Carla got:

$$\underbrace{12}_{Alice's\ bid} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = -2\ left$$

Bob uses his own bid when trying to compute Carla's compensation. Bob thinks Carla got:

$$\underbrace{21}_{Alice's\ bid} - \underbrace{7}_{To\ Alice} - \underbrace{7}_{To\ Bob} = 7\ left$$

Both Alice and Bob think Carla got less than or equal to their own compensations, due to their lower bids. They do not envy Carla. Alice even thinks Carla got ripped off and lost money.

No Envy (Alice)

Name	Bids	Fair Share	Compensation
Alice	12	4	7 from Carla
Bob	21	7	7 from Carla
Carla	24	8	10 left

Alice's (actual) compensation is

$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

When Bob and Carla try to compute Alice's compensation, they will think Alice got

$$\underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

This is the same as Alice's actual compensation, since she has no item whose value is not agreed upon.

Alice's (actual) compensation is

$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

When Bob and Carla try to compute Alice's compensation, they will think Alice got

$$\underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

This is the same as Alice's actual compensation, since she has no item whose value is not agreed upon.
Carla is okay, since her own compensation is higher than Alice's. Carla felt that she did not pay Alice too much and that she kept enough for her own compensation.
Bob is okay, since his compensation is the same as Alice's. He is fine with being paid the same as her.

No Envy (Bob)

Name	Bids	Fair Share	Compensation
Alice	12	4	7 from Carla
Bob	21	7	7 from Carla
Carla	24	8	10 left

Bob's (actual) compensation is

$$x_{Bob} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

When Alice and Carla try to compute Bob's compensation, they will think Bob got

$$\underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

This is the same as Bob's actual compensation, since he has no item whose value is not agreed upon.

Bob's (actual) compensation is

$$x_{Bob} = \underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

When Alice and Carla try to compute Bob's compensation, they will think Bob got

$$\underbrace{0}_{No\ Item} + \underbrace{7}_{From\ Carla} = 7\ from\ Carla$$

This is the same as Bob's actual compensation, since he has no item whose value is not agreed upon.
Carla is okay, since her own compensation is higher than Bob's. Carla felt that she did not pay Bob too much and that she kept enough for her own compensation.
Alice is okay, since her compensation is the same as Bob's. She is fine with being paid the same as him.

Envy-free Arrangement (Alternate Solution)

Name	Bids	Fair Share	Compensation
Alice	12	4	8 from Carla
Bob	21	7	8 from Carla
Carla	24	8	

We saw that Carla should pay Alice and Bob between 7 and 8 dollars.
For this example, let us have her pay 8 each, the maximum to Alice and Bob.
Their compensations will be what Carla pays them, since they have no item:

$$x_{Alice} = \underbrace{0}_{No\ Item} + \underbrace{8}_{From\ Carla} = 8\ from\ Carla$$

$$x_{Bob} = \underbrace{0}_{No\ Item} + \underbrace{8}_{From\ Carla} = 8\ from\ Carla$$

Name	Bids	Fair Share	Compensation
Alice	12	4	8 from Carla
Bob	21	7	8 from Carla
Carla	24	8	8 ***left***

Carla is not going to pay herself 8 dollars!

- Carla has the item, which is worth 24 to her.
- Carla lost 8 when paying Alice, and lost 8 when paying Bob.
- Thus, Carla's compensation is what she has left after paying the others:

$$x_{Carla} = \underbrace{24}_{Item} - \underbrace{8}_{To\ Alice} - \underbrace{8}_{To\ Bob} = 8\ left$$

Carla's compensation is the same number as Alice and Bob's compensations, but it came about in a different way. Alice and Bob were *paid* 8 dollars each, while Carla is *left* with 8 dollars after paying.

Next time

- Exam 3