Symmetry Comments

Now that you have spent a week twirling cube structures, I thought I would add some concluding remarks. Some objectives of this exercise, which is a direct extension of one of the units in *Ruins of Montarek* in the *Connected Mathematics* curriculum, are to better understand three-dimensional symmetry, as well as representation of three-dimension objects in two-dimensional diagrams.

As you saw from the first question, there are 24 ways to pick up a cube and replace it in its initial position (say, on a table). One way to argue this is to note that there are 6 choices for the face that will be upright, and then 4 choices for vertex that will be facing you in the isometric drawing, for a total of 24 choices. Thus the cube has 24 direct symmetries (direct because we are not including reflections—if reflections are also allowed, then there are an additional 24 indirect symmetries for a total of 48). We can distinguish these 24 ways if the faces of the cube are colored. In this case there are 24 isometric drawings of a single cube from above, and another 24 from below, for a total of 48. However, once the faces are blank, the cube's symmetries cause the 24 orientations to become indistinguishable, giving a total of one distinct view from above and one from below.

In general, then, for a cubical structure there are 24 orientations in which the structure can be positioned in space in front of you in preparation for an isometric drawing—just focus on one particular "reference cube" and its 24 orientations. This results in potentially 48 different drawings—each orientation can be drawn from either above or below. Now if the structure possesses some symmetry, applying a symmetry to a particular orientation will lead to another orientation with the same drawing. If there are n symmetries, then the n orientations resulting from applying these symmetries to a particular orientation will result in n indistinguishable drawings. These n orientations derived from a specific one by applying symmetries fall into a family of size n. Now, since there are 24 total possible orientations, and they fall into families of indistinguisable orientations, each of size n, we conclude that there are $24/n$ families, each with a unique drawing from above, and another from below, resulting in $48/n$ different drawings on isometric dot paper.

- For a cube, $n = 24$, so there are $48/24 = 2$ drawings.
- For a pair of cubes, $n = 8$, so there are $48/8 = 6$ drawings.
- For a three-cube "L", $n = 2$, so there are $48/2 = 24$ drawings.
- For a four-cube "L", $n = 1$ (remember, we are not including reflections), so there are $48/1 = 48$ drawings.

- For the six-cube structure consisting of a "plus-sign" plus an extra cube attached to the center, for example, $n = 4$, so there are $48/4 = 12$ drawings.
- Removing two opposite corners from a $2 \times 2 \times 2$ block, for example, results in $n = 6$, so there are $48/6 = 8$ drawings.
- Removing one corner from a $2 \times 2 \times 2$ block, for example, results in $n = 3$, so there are $48/3 = 16$ drawings.
- Removing four alternate corners from a $2 \times 2 \times 2$ or $3 \times 3 \times 3$ block, for example, results in $n = 12$, so there are $48/12 = 4$ drawings.