THE AMER-BRUMER THEOREM OVER ARBITRARY FIELDS

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Abstract. The Amer-Brumer Theorem was originally proved for fields with characteristic different from 2. Amer's generalization of Brumer's result has never been published. We show that Amer's result extends to fields of arbitrary characteristic and we give a proof that is independent of the characteristic of the field.

1. Introduction and preliminaries

Brumer published a beautiful theorem in [B] stating that two quadratic forms Q_1, Q_2 defined over a field k, $\operatorname{char} k \neq 2$, have a common nontrivial zero over k if and only if the quadratic form $Q_1 + tQ_2$ has a nontrivial zero over the rational function field $k(t)$. Brumer's proof used a very clever argument involving the use of hyperplane reflections. Independently and slightly earlier, Amer proved a stronger version of this result in [A]. He showed the same result holds for r-dimensional subspaces of zeros. Amer also restricted to fields k with $\operatorname{char} k \neq 2$. Amer's proof has never been published. Since the result is so fundamental, it seems worthwhile to publish a proof. The proof given below (Theorem 2.2) has been extended to cover all fields. The proof is also a bit simpler than Amer's original proof in that hyperplane reflections are used more efficiently. The Amer-Brumer Theorem will refer to the case $r = 1$ in Theorem 2.2 and Amer's Theorem will refer to the general case. Pfister published a proof of the Amer-Brumer Theorem in [P], page 137.

Let k be a field, let V be a finite dimensional vector space over k and let $Q : V \to k$ be a quadratic map. This means that $Q(av) = a^2Q(v)$ for all $v \in V$, $a \in k$, and the function $B : V \times V \to k$ defined by $B(v, w) = Q(v + w) - Q(v) - Q(w)$ for all $v, w \in V$ is a symmetric bilinear form. Thus $B(v, v) = 2Q(v)$. We let $\operatorname{rad} B = \{v \in V | B(v, w) = 0 \text{ for all } w \in V\}$ and we let $\operatorname{rad} Q = \{v \in V | Q(v + w) = Q(w) \text{ for all } w \in V\}$. If $v \in \operatorname{rad} Q$, then $Q(v) = 0$ (set $w = 0$) and it follows that $\operatorname{rad} Q \subseteq \operatorname{rad} B$. An isomorphic linear transformation

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$\sigma : V \to V$ is an isometry of (V, Q) if $Q(v) = Q(\sigma(v))$ for all $v \in V$. If σ is an isometry, it follows that $B(v, w) = B(\sigma(v), \sigma(w))$ for all $v, w \in V$. The following lemma introduces some special isometries that are often called hyperplane reflections when $\operatorname{char} k \neq 2$ and orthogonal transvections when $\operatorname{char} k = 2$.

Lemma 1.1. *Let $z \in V$ and assume $Q(z) \neq 0$. Let $\tau_z : V \to V$ be defined by*

$$\tau_z(y) = y - \frac{B(y,z)}{Q(z)} z.$$

Then the following statements hold.

(1) *τ_z is a linear transformation.*
(2) *$\tau_z \circ \tau_z = 1$ and $\tau_z(z) = -z$.*
(3) *$Q(y) = Q(\tau_z(y))$ for all $y \in V$. Thus τ_z is an isometry of (V, Q).*

Proof. These are straightforward calculations. □

The following result is needed in the proof of Theorem 2.2.

Lemma 1.2. *Let $Q : V \to k$ be a quadratic map with $B : V \times V \to k$ the associated symmetric bilinear form. Then every maximal subspace of V on which Q vanishes has the same dimension.*

Proof. When B is nonsingular, this result is well known and the dimension h of a maximal subspace of V on which Q vanishes is often called the Witt index of Q. In the general case the dimension of such a maximal subspace of V is $h + \dim(\operatorname{rad} Q)$, where h is the Witt index of the nonsingular part of Q. □

2. Amer's Theorem for arbitrary k

Let $Q_1 : V \to k$ and $Q_2 : V \to k$ be two quadratic maps and let B_1, B_2 be the associated symmetric bilinear forms. Let $k(t)$ denote the rational function field and let $V_{k(t)} = V \otimes_k k(t)$. Let

$$Q_1 + tQ_2 : V_{k(t)} \to k(t)$$

be the function defined by $(Q_1 + tQ_2)(v) = Q_1(v) + tQ_2(v)$, where $v \in V_{k(t)}$. It is easily checked that $Q_1 + tQ_2$ is a quadratic map. The associated symmetric bilinear form is denoted $B_1 + tB_2$ and satisfies $(B_1 + tB_2)(v, w) = B_1(v, w) + tB_2(v, w)$. We make no assumptions on Q_1, Q_2, B_1, B_2 concerning nondegeneracy. Let

$$v = v_0 + tv_1 + \cdots + t^n v_n \in V_{k[t]} = V \otimes_k k[t],\ v_i \in V.$$

If $v_n \neq 0$, we say that $\deg v = n$. The following lemma was proved by Brumer [B] when $\operatorname{char} k \neq 2$.

Lemma 2.1. *Suppose $(Q_1+tQ_2)(v)=0$ where $v=v_0+tv_1+\cdots+t^nv_n$, $v_i\in V$. Assume that $(Q_1+tQ_2)(v_n)\neq 0$ and let $v'=\tau_{v_n}(v)$, where τ_{v_n} is the isometry of $(V_{k(t)},Q_1+tQ_2)$ defined in Lemma 1.1. Then $v'\in V_{k[t]}$ and $\deg v'<\deg v$.*

Proof. Since $(Q_1+tQ_2)(v)=0$, a computation of the coefficients of t^{2n+1} and t^{2n} shows that

(1) $Q_2(v_n)=0$,
(2) $Q_1(v_n)+B_2(v_{n-1},v_n)=0$.

Since $(Q_1+tQ_2)(v_n)\neq 0$ and $Q_2(v_n)=0$ by (1), we have $Q_1(v_n)\neq 0$. This gives

$$v'=\tau_{v_n}(v)=v-\frac{(B_1+tB_2)(v,v_n)}{(Q_1+tQ_2)(v_n)}v_n=v-\frac{B_1(v,v_n)+tB_2(v,v_n)}{Q_1(v_n)}v_n.$$

Thus $v'=v_0'+tv_1'+\cdots+t^{n-1}v_{n-1}'+t^nv_n'+t^{n+1}v_{n+1}'$, where $v_i'\in V$. We have

$$v_{n+1}'=-\frac{B_2(v_n,v_n)}{Q_1(v_n)}v_n=0,$$

since $B_2(v_n,v_n)=2Q_2(v_n)=0$. We have

$$\begin{aligned}v_n'&=v_n-\frac{B_1(v_n,v_n)+B_2(v_{n-1},v_n)}{Q_1(v_n)}v_n\\&=\frac{Q_1(v_n)-B_1(v_n,v_n)-B_2(v_{n-1},v_n)}{Q_1(v_n)}v_n=0\end{aligned}$$

using $Q_1(v_n)-B_1(v_n,v_n)=-Q_1(v_n)$ and (2). □

The following theorem was proved by Amer in [A] when $\operatorname{char} k\neq 2$. The proof given here is a slight simplification of Amer's proof and is valid for all fields.

Theorem 2.2. *Let Q_1, Q_2 be as above. Then Q_1, Q_2 vanish on a common r-dimensional subspace of V if and only if Q_1+tQ_2 vanishes on an r-dimensional subspace of $V_{k(t)}$.*

Proof. If Q_1, Q_2 vanish on a common r-dimensional subspace W of V, then Q_1+tQ_2 vanishes on $W_{k(t)}\subseteq V_{k(t)}$.

Assume Q_1+tQ_2 vanishes on an r-dimensional subspace W of $V_{k(t)}$. Let $\{w_1,\dots,w_r\}$ be a basis of W. We may assume each $w_i\in V_{k[t]}=V\otimes_k k[t]$. Thus we may write $w_i=v_{i0}+tv_{i1}+\cdots+t^{n_i}v_{i,n_i}$, where each $v_{ij}\in V$ and $v_{i,n_i}\neq 0$, so that $\deg w_i=n_i$. We may arrange the w_i's so that $n_1\leq n_2\leq\cdots\leq n_r$.

Among all r-dimensional subspaces W of $V_{k(t)}$ on which Q_1+tQ_2 vanishes, and among all possible bases $\{w_1,\dots,w_r\}$ of these subspaces W such that $w_i\in V_{k[t]}$, $1\leq i\leq r$, and $n_1\leq n_2\leq\cdots\leq n_r$, choose a

subspace W and a basis $\{w_1, \ldots, w_r\}$ such that $(n_1, \ldots, n_r)$ is minimal with respect to the lexicographical ordering. Our goal is to show that $n_1 = \cdots = n_r = 0$. Suppose this is not the case. Then we may assume $\deg w_1 = \cdots = \deg w_{i-1} = 0$ and $\deg w_i > 0$ for some $i \geq 1$. For ease of notation, set $n = n_i$ and $v_j = v_{ij}$, $0 \leq j \leq n_i = n$. Then $w_i = v_0 + tv_1 + \cdots + t^n v_n$ with $v_n \neq 0$ and $n \geq 1$.
Case I. Suppose $Q_1 + tQ_2$ does not vanish on the k-subspace generated by $\{w_1, \ldots, w_{i-1}, v_n\}$. Then $(Q_1 + tQ_2)(y + v_n) \neq 0$ for some $y \in \operatorname{span}_k\{w_1, \ldots, w_{i-1}\}$. Since W is generated as a $k(t)$-subspace by $\{w_1, \ldots, w_{i-1}, w_i + t^n y, w_{i+1}, \ldots, w_r\}$, and since

$$w_i + t^n y = v_0 + tv_1 + \cdots + t^{n-1}v_{n-1} + t^n(y + v_n),$$

we may assume from the start that $w_i = v_0 + tv_1 + \cdots + t^n v_n$ with $(Q_1 + tQ_2)(v_n) \neq 0$. Since $(Q_1 + tQ_2)(w_i) = 0$, we have $Q_2(v_n) = 0$. Thus $Q_1(v_n) \neq 0$.

Let τ_{v_n} be the isometry of $(V_{k(t)}, Q_1 + tQ_2)$ defined in Lemma 1.1. Let

$$Y = \tau_{v_n}(W) = \operatorname{span}_{k(t)}\{\tau_{v_n}(w_1), \ldots, \tau_{v_n}(w_r)\}.$$

Then $\tau_{v_n}(w_i) \in V_{k[t]}$ and $\deg(\tau_{v_n}(w_i)) < \deg(w_i)$ by Lemma 2.1. We have that $(B_1 + tB_2)(w_j, w_i) = 0$, since $Q_1 + tQ_2$ vanishes on W. The coefficient of t^{n+1} in $(B_1 + tB_2)(w_j, w_i)$ equals $B_2(w_j, v_n)$, and thus $B_2(w_j, v_n) = 0$. Since $Q_2(v_n) = 0$ and $Q_1(v_n) \neq 0$, we now have for $1 \leq j < i$ that

$$\tau_{v_n}(w_j) = w_j - \frac{(B_1 + tB_2)(w_j, v_n)}{(Q_1 + tQ_2)(v_n)} v_n = w_j - \frac{B_1(w_j, v_n)}{Q_1(v_n)} v_n \in V.$$

After rearranging this basis of Y lexicographically by degree, we produce a smaller r-tuple of degrees, which is a contradiction.
Case II. Suppose $Q_1 + tQ_2$ vanishes on the k- subspace generated by $\{w_1, \ldots, w_{i-1}, v_n\}$. If $\{w_1, \ldots, w_{i-1}, v_n\}$ is a linearly dependent set over k, then there exists $y \in \operatorname{span}_k\{w_1, \ldots, w_{i-1}\}$ such that $y + v_n = 0$. Then

$$\begin{aligned}&\operatorname{span}_{k(t)}\{w_1, \ldots, w_i, \ldots, w_r\}\\ &\qquad = \operatorname{span}_{k(t)}\{w_1, \ldots, w_{i-1}, t^n y + w_i, w_{i+1}, \ldots, w_r\}.\end{aligned}$$

But $t^n y + w_i \neq 0$ since $\{w_1, \ldots, w_i\}$ are linearly independent, and $\deg(t^n y + w_i) < \deg w_i$ since $y + v_n = 0$. This basis of W produces a smaller r-tuple of degrees, which is a contradiction.

Thus $\{w_1, \ldots, w_{i-1}, v_n\}$ is a linearly independent set over k. Then Lemma 1.2 implies that $\{w_1, \ldots, w_{i-1}, v_n\}$ is contained in a subspace Y of $V_{k(t)}$, with $\dim Y = r$, on which $Q_1 + tQ_2$ vanishes. Then Y has a basis that gives a smaller r-tuple of degrees, our final contradiction.

Therefore $n_1 = \cdots = n_r = 0$ and it follows that $Q_1 + tQ_2$ vanishes on an r-dimensional subspace W of V. Then Q_1 and Q_2 vanish simultaneously on the r-dimensional subspace W of V. $\square$

3. An application

In this section we use the Amer-Brumer Theorem to give a new proof in Proposition 3.1 of one of the representation theorems due to Cassels and Pfister (see [L], p. 260 or [S], p. 150). In Proposition 3.2, we use Amer's Theorem to reprove Proposition 3.1 and in fact prove a slightly stronger version.

Let k be a field with $\operatorname{char} k \neq 2$ and let $k(t)$ denote the rational function field. If q is a quadratic form defined over k, let $D_k(q)$ denote the nonzero values of k represented by q.

In Propositions 3.1 and 3.2, let $q \in k[x_1, \ldots, x_n]$ be an anisotropic quadratic form.

Proposition 3.1. *Suppose $q \cong \langle b \rangle \perp \phi$ and let $d \in k$, $d \neq 0$. Then $d \in D_k(\phi)$ if and only if $bt^2 + d \in D_{k(t)}(q)$.*

Proof. Without loss of generality, we can assume that $b = 1$ by scaling q and d by b. If $d \in D_k(\phi)$, then it is clear that $t^2 + d \in D_{k(t)}(q)$. Now assume $t^2 + d \in D_{k(t)}(q)$. We note that

$$\langle 1, -(t^2+d) \rangle \cong \begin{pmatrix} 1 & t \\ t & -d \end{pmatrix},$$

since both binary forms represent 1 over $k(t)$ and both have determinant $-(t^2 + d)$. Let

$$Q_1 = \phi(x_1, \ldots, x_{n-1}) + x_n^2 - dx_{n+1}^2 \tag{1}$$

$$Q_2 = 2x_n x_{n+1}. \tag{2}$$

Then

$$Q_1 + tQ_2 \cong_{k(t)} \phi \perp \begin{pmatrix} 1 & t \\ t & -d \end{pmatrix} \tag{3}$$

$$\cong_{k(t)} \phi \perp \langle 1, -(t^2+d) \rangle \cong_{k(t)} q \perp \langle -(t^2+d) \rangle. \tag{4}$$

Therefore $Q_1 + tQ_2$ is isotropic over $k(t)$. The Amer-Brumer Theorem implies that Q_1, Q_2 have a common nontrivial zero over k. Then either $x_n = 0$ or $x_{n+1} = 0$. If $x_{n+1} = 0$, then q is isotropic over k, a contradiction. Thus $x_n = 0$ and $\phi \perp \langle -d \rangle$ is isotropic over k. Thus $d \in D_k(\phi)$. $\square$

In the next proposition, we weaken the hypothesis by not assuming beforehand that b is represented by q over k. This stronger version requires Amer's Theorem. In the usual treatment (see [L] or [S], for example), an application of the so-called substitution principle is usually used.

Proposition 3.2. *Let $b, c \in k$ with $bc \neq 0$. Then $\langle b, c\rangle$ is a k-subform of q if and only if $bt^2 + c \in D_{k(t)}(q)$.*

Proof. If $\langle b, c\rangle$ is a k-subform of q, then it is clear that $bt^2+c \in D_{k(t)}(q)$. Now suppose that $bt^2 + c \in D_{k(t)}(q)$. Let $d = cb^{-1}$ so that $bt^2 + c = b(t^2 + d)$. We have $q \perp \langle -b(t^2 + d)\rangle$ is isotropic over $k(t)$ and hence $q \perp \langle -b, b, -b(t^2 + d)\rangle$ vanishes on a 2-dimensional vector space over $k(t)$. We note that

$$\langle b, -b(t^2 + d)\rangle \cong_{k(t)} b \begin{pmatrix} 1 & t \\ t & -d \end{pmatrix},$$

since both binary forms represent b over $k(t)$ and both have determinant $-(t^2 + d)$. Let

$$\begin{aligned} Q_1 &= q(x_1, \ldots, x_n) - bx_{n+1}^2 + bx_{n+2}^2 - bdx_{n+3}^2 \\ Q_2 &= \qquad\qquad\qquad\qquad\qquad 2bx_{n+2}x_{n+3}. \end{aligned}$$

Then

$$\begin{aligned} Q_1 + tQ_2 &\cong_{k(t)} q \perp \langle -b\rangle \perp b \begin{pmatrix} 1 & t \\ t & -d \end{pmatrix} \\ &\cong_{k(t)} q \perp \langle -b, b, -b(t^2 + d)\rangle. \end{aligned}$$

Thus $Q_1 + tQ_2$ vanishes on a 2-dimensional vector space over $k(t)$. Amer's Theorem (Theorem 2.2) implies Q_1, Q_2 vanish on a common 2-dimensional k-subspace W defined over k. Then W is contained in the union of the two hyperplanes defined by $\{x_{n+2} = 0\} \cup \{x_{n+3} = 0\}$. It follows easily that either $W \subseteq \{x_{n+2} = 0\}$ or $W \subseteq \{x_{n+3} = 0\}$. If $W \subseteq \{x_{n+3} = 0\}$, then $q(x_1, \ldots, x_n) - bx_{n+1}^2 + bx_{n+2}^2$ vanishes on a 2-dimensional subspace over k and so $q(x_1, \ldots, x_n)$ must be isotropic over k, a contradiction. Therefore $W \subseteq \{x_{n+2} = 0\}$ and so $q(x_1, \ldots, x_n) - bx_{n+1}^2 - bdx_{n+3}^2$ vanishes on a 2-dimensional subspace over k. Then

$$\begin{aligned} q \perp \langle -b, -bd\rangle &\cong_k q' \perp \mathbb{H} \perp \mathbb{H} \\ &\cong_k q' \perp \langle b, bd, -b, -bd\rangle \end{aligned}$$

for some quadratic form q'. Witt cancellation implies that $q \cong_k q' \perp \langle b, bd\rangle$ and so $\langle b, bd\rangle = \langle b, c\rangle$ is a k-subform of q. $\square$

References

[A] M. Amer, *Quadratische Formen über Funktionenkörpern*, unpublished dissertation, Johannes Gutenberg Universität, Mainz (1976).

[B] A. Brumer, *Remarques sur les couples de formes quadratiques*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 16, A679–A681.

[L] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Reading, Mass., (1980).

[P] A. Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, Cambridge University Press, (1995).

[S] W. Scharlau, *Quadratic and Hermitian Forms*, Springer, Berlin, (1985).

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