MA 113 WORKSHEETS
SPRING 2024

There are three goals for students in MA 113 recitations:

1. to develop your ability to make sense of problems and be persistent while solving them,
2. to develop your ability to productively collaborate with peers, and
3. to develop your ability to check your own work, i.e. to decide on your own whether or not your work is correct.

To help you reach these goals, you will spend the majority of the recitation working in small groups on worksheets. You may not be able to complete all of these problems — your TA will help guide you in selecting which problems to work on first. Your focus should be to discover your misunderstandings by doing math collaboratively. Mistakes and misunderstandings are a normal part of learning mathematics — the only path to deep learning is to learn to effectively identify and revise our mistakes and misunderstandings. Recitations are not intended to be homework help sessions. If you have questions about the homework, you may speak with one of your instructors outside of class, send an email from WeBWorK, or visit the Mathskeller or the Study.

Solutions to MA 113 worksheets are not provided. Instead, you should focus on using these problems to test your self-evaluation skills. Imagine you are taking an exam, and you need to check for yourself whether or not your work is correct — this is a skill you need to practice in order to do well! By collaborating with your peers and comparing solutions, with guidance and support from your TA, your problem solving and self-evaluation skills will improve. If there are worksheet problems that you are uncertain about, you are welcome to ask about them during recitation, during your TA or instructor office hours, at the Mathskeller, or at the Study.

Contents

§0.4 Functions

1. Find the domain and range of the following functions.

   (a) $a(x) = 4x - 3$

   (b) $b(x) = \dfrac{x+1}{x-1}$

   (c) $c(x) = \sqrt{5 + 4x - x^2}$

   (d) $d(x) = \dfrac{1}{\sqrt{5-x}}$

2. Suppose that $f(x)$ has domain $[3, 15]$ and range $[-2, 8]$. What are the domain and range of $f(x + 7)$ and $3f(x)$?

3. If$f(x) = 2x + 3$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. Are the functions $f \circ g$ and $g \circ f$ the same function? Why or why not?

4. Let $f(x) = x^2$. Simplify the following expressions.

   (a) $f(3 + 2h)$

   (b) $\dfrac{f(2+h) - f(2-h)}{h}$

   (c) $\dfrac{f(a+h) - f(a)}{h}$

5. Let $f(x) = x + 3$ and consider the function h_n obtained by composing n copies of f. So $h_2 = f \circ f$ and $h_n = f \circ h_{n-1}$. Compute h_n for several values of n. Can you guess a simple formula for $h_n(x)$?

6. Write the function $f(x) = 2x^2 + 3$ as a composition of two simpler functions $f = g \circ h$. Can you find more than one answer?

7. A ball is thrown in the air from ground level. The height of the ball in meters at time t seconds is given by the function $h(t) = -4.9t^2 + 20t$. At what time does the ball hit the ground? (Be sure to use the proper units!)

8. A rectangular pen is to be constructed using 40 meters of fencing. Write a function which gives the area enclosed by the pen in terms of the length x of one of the sides. Use that x represents a side length to determine the domain of this function. Explain how you found the domain.

9. A box has a square base of side s meters and total volume of 300 square meters. Write a function A which gives surface area of the box as a function of s. What is the domain of the function? Why?

§0.6 Inverse functions

1. Determine which of the following functions are one to one.

   (a) $a(x) = x^2$

   (b) $b(x) = \dfrac{1}{(x+2)}$

   (c) $c(x) = \cos(x)$

   (d) $d(x) = |x+2|$

2. Find the inverse of the following functions. Give the domain and range of each inverse function.

   (a) $f(x) = 2x - 6$

   (b) $g(x) = \dfrac{x+2}{2x-1}$

3. Let $f(x) = mx+b$ and find the inverse function f^{-1}, if possible. What condition do you need for the inverse to exist? What do you notice about the slope of the inverse function?

4. Let $f(x) = x^2 - 2x$.

   (a) Find the largest interval of the form $(-\infty, b]$ for which f is one to one.

   (b) Let g be function given by the formula $x^2 - 2x$ and with domain the interval you found in part a). Find the inverse function g^{-1}.

   (c) Sketch the graphs of g and g^{-1} and use the graph to confirm that your answer to b) is correct.

   (d) Check your answer in part b) by computing $g \circ g^{-1}$.

5. Write the function $f(x) = 2x + 4$ as a composition $f = m \circ t$ of a translation $t(x) = x + a$ and a dilation or multiplication $m(x) = bx$. Find the inverse functions of m and t and write the inverse f^{-1} as a composition of the inverses of m and t. Be sure to check your answer. What do you notice?

6. If c is the temperature in degrees Celsius and f is the temperature in degrees Fahrenheit then we have $f = 1.8c + 32$. Find a function $T(f)$ which gives converts the temperature in degrees Fahrenheit to the corresponding temperature in degrees Celsius.

§§1.1-2 Tangent lines and average velocity

1. Find the equations of the following lines.

   (a) The line passing through $(2,5)$ and $(-1,-1)$.

   (b) The line passing through $(2,4)$ with slope -3.

   (c) The line with slope 2 and y-intercept 5.

   (d) The line passing through $(3,4)$ and perpendicular to $y=-2x+1$

2. Suppose that the equation of the tangent line to the curve $y=f(x)$ at $x=3$ is $y=4x-3$. Find $f(3)$ and the slope of the graph of f at $x=3$.

3. Let $f(x)=x^2$.

   (a) Find the slope of the secant line which passes through the points $(3,f(3))$ and $(4,f(4))$.

   (b) Find the slope of the secant line which passes through $(3,f(3))$ and $(3.1,f(3.1))$.

   (c) Find the slope of the secant line which passes through $(3,f(3))$ and $(3.01,f(3.01))$.

   (d) Guess the slope of the tangent line to the graph of $f(x)=x^2$ at $x=3$.

4. Let $g(x)=1/x$.

   (a) Find the slope of the secant line which passes through the points $(2,g(2))$ and $(2.1,g(2.1))$.

   (b) Find the slope of the secant line which passes through $(2,g(2))$ and $(2+h,g(2+h))$.

   (c) Simplify your answer in part b) to obtain an expression which is defined if $h=0$.

   (d) Guess the slope of the tangent line to the graph of $g(x)=1/x$ at $x=2$.

5. Compute the slope of the lines through $(0,\sin(0))$ and $(h,\sin(h))$ for several small values of h to estimate the slope of the curve $y=\sin(x)$ at $x=0$.

6. The height in meters of an object at time t seconds is $h(t)=-5t^2+20t$. Find the average velocity of the object on the interval $2\le t\le 2.5$.

7. Let $p(t)=t^3-45t$ denote the distance (in meters) to the right of the origin of a particle at time t minutes after noon.

   (a) Find the average velocity of the particle on the intervals $[2,2.1]$ and $[2,2.01]$.

(b) Use this information to guess a value for the instantaneous velocity of particle at 12:02pm.

8. A particle is moving along a straight line so that its position at time t seconds is given by $s(t) = 4t^2 - t$ meters.

   (a) Find the average velocity of the particle over the time interval $[1, 2]$.

   (b) Determine the average velocity of the particle over the time interval $[2, t]$ where $t > 2$. Simplify your answer. [Hint: Factor the numerator.]

   (c) Based on your answer in (b) can you guess a value for the instantaneous velocity of the particle at $t = 2$?

§1.3 The limit of a function

1. Compute the value of $\dfrac{e^z - 1}{z}$ for $z = 0.1, -0.02, 0.005$ and a few more small values. Based on your results, guess the value of the limit $\lim\limits_{z \to 0} \dfrac{e^z - 1}{z}$.

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x - 1 & \text{if } 0 < x \text{ and } x \neq 2 \\ -3 & \text{if } x = 2 \end{cases}$.

   (a) Sketch the graph of f.

   (b) Compute the following:

i. $\lim\limits_{x \to 0^-} f(x)$	iii. $\lim\limits_{x \to 0} f(x)$	v. $\lim\limits_{x \to 2^-} f(x)$	vii. $\lim\limits_{x \to 2} f(x)$
ii. $\lim\limits_{x \to 0^+} f(x)$	iv. $f(0)$	vi. $\lim\limits_{x \to 2^+} f(x)$	viii. $f(2)$

3. Determine if the following state is correct and give an example to support your answer. If $\lim_{x \to 4} f(x) = 42$, then $f(4) = 42$.

4. Let $h(x) = \begin{cases} 2x + b, & x < -1 \\ x^2, & x > -1 \end{cases}$ Can you find b so that $\lim\limits_{x \to -1} h(x)$ exists?

5. Let $h(x) = \frac{|x|}{x}$. Give the domain of h. For which value of a does the limit $\lim\limits_{x \to a} h(x)$ not exist. Do the one-sided limits exist at this point?

6. Sketch the graphs of $f(x) = 1/(x - 5)$ and $g(x) = 1/(x + 5)^2$. Find the limits or state that they do not exist.

   (a) $\lim\limits_{x \to 5^+} \dfrac{1}{x - 5}$

   (b) $\lim\limits_{x \to 5^-} \dfrac{1}{x - 5}$

   (c) $\lim\limits_{x \to 5} \dfrac{1}{x - 5}$

   (d) $\lim\limits_{x \to -5^+} \dfrac{1}{(x + 5)^2}$

   (e) $\lim\limits_{x \to -5^-} \dfrac{1}{(x + 5)^2}$

   (f) $\lim\limits_{x \to -5} \dfrac{1}{(x + 5)^2}$

§1.4 Limit laws

Some history: Mathematicians did not give a formal theory of limits between the invention of calculus in the 1660's and the formal definition of a limit in the 1820's. Even after the 1820's, mathematicians and scientists wrote lim without writing $x \to a$ below it. It appears that the widespread use of the notation $\lim\limits_{x\to a}$ was only adopted in the early 1900's after being used in several books, including one by G. H. Hardy titled "A Course of Pure Mathematics."

1. Starting with the basic limits $\lim\limits_{x\to a} x = a$ and for a constant c, $\lim\limits_{x\to a} c = c$, use the limit laws to find the following limits.

   (a) $\lim\limits_{x\to 2} x^2 + 2x$

   (b) $\lim\limits_{x\to 2} x + 3x^2 + 2x$

2. Review the rule for the limit of quotient in Theorem 1.4.3 and determine which of the following limits can be evaluated using this rule (without first simplifying). If the quotient rule applies, find the limit.

   (a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2+4}$

   (b) $\lim\limits_{x\to 2} \dfrac{x^2+4}{x^2-4}$

   (c) $\lim\limits_{x\to 1} \dfrac{x}{x^3-1}$

   (d) $\lim\limits_{x\to 1} \dfrac{x-1}{x^3+1}$

3. Given $\lim\limits_{x\to 2} f(x) = 5$ and $\lim\limits_{x\to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot use the limit laws to find the limit.

   (a) $\lim\limits_{x\to 2}(2f(x) - g(x))$

   (b) $\lim\limits_{x\to 2} f(x)^2 + x \cdot g(x)^2$

   (c) $\lim\limits_{x\to 2} \dfrac{f(x)-5}{g(x)-2}$

   (d) $\lim\limits_{x\to 2} \dfrac{f(x)g(x)}{x}$

   (e) $\lim\limits_{x\to 2}(f(x)g(2))$

   (f) $\lim\limits_{x\to 2}[f(x)]^{\frac{3}{2}}$

4. Let $f(x) = 1 + x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$. Consider $\lim\limits_{x\to 0} f(x)$.

   (a) Find two simpler functions, g and h, that satisfy the hypothesis of the Squeeze Theorem (Theorem 1.4.18).

   (b) Determine $\lim\limits_{x\to 0} f(x)$ using the Squeeze Theorem.

   (c) Use a calculator to produce a graph that illustrates this application of the Squeeze Theorem.

5. Suppose $f(x) \leq g(x) \leq h(x)$ with $f(x) = x$ and $h(x) = x^2 - x + 1$. There is one point a where we may use the squeeze theorem to find the value of $L = \lim_{x \to a} g(x)$. Find a and then L.

6. Suppose that a particle is moving along a line. The position of particle at time t seconds is $p(t) = t^3 - 2t$ meters to the right of the origin.

   (a) Find the instantaneous velocity at time $t = 3$ seconds.

   (b) Is the particle moving to the left or the right at $t = 3$.

§1.5 Limits at infinity (part 1)

1. Evaluate the following limits, or explain why the limit does not exist:

   (a) $\lim_{x\to\infty} \dfrac{3x^2 - 7x}{x - 8}$

   (b) $\lim_{x\to\infty} \dfrac{2x^2 - 6}{x^4 - 8x + 9}$

   (c) $\lim_{x\to-\infty} \cos(x)$

   (d) $\lim_{x\to-\infty} 3$

   (e) $\lim_{x\to\pm\infty} \dfrac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$

   (f) $\lim_{x\to-\infty} \dfrac{\sqrt{x^{10} + 2x}}{x^5}$

2. Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.

§1.5 Limits at infinity (part 2)

1. Explain the difference between "$\lim_{x\to -3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".

2. Suppose that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Find examples of f and g so that the following statements fail and (obviously different) examples so that the statements hold true.

   (a) $\lim_{x\to\infty} (f(x)/g(x)) = 1$.

   (b) $\lim_{x\to\infty} (f(x) - g(x)) = 0$.

3. Find the limit $\lim_{t\to\infty} (\sqrt{t+1} - \sqrt{t+2})$. Hint: Remember that sometimes expressions of the form $\sqrt{a} - \sqrt{b}$ may be simplified by multiplying by the expression

$$1 = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$$

   In other words, by multiplying and dividing by the conjugate of $\sqrt{a} - \sqrt{b}$.

4. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ if $f(x) = \left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right)$.

5. Sketch a graph with all of the following properties:

   - $\lim_{t\to\infty} f(t) = 2$
   - $\lim_{t\to -\infty} f(t) = 0$
   - $\lim_{t\to 0^+} f(t) = \infty$
   - $\lim_{t\to 0^-} f(t) = -\infty$
   - $\lim_{t\to 4} f(t) = 3$
   - $f(4) = 6$

§1.6 Continuity

1. State the definition of "the function f is continuous at a" and list the three things you must verify to show a function f is continuous at a.

2. Decide if each of the following statements are true or false. If a statement is false, provide a specific example of a function $f(x)$ that supports your answer.

   (a) Every function is continuous on its domain.

   (b) If $\lim_{x\to a} f(x)$ exists, then f is continuous at a.

   (c) If f is continuous at a, then $\lim_{x\to a} f(x)$ exists.

   (d) If f is continuous at $x = a$, then $\lim_{x\to a^+} f(x)$ exists.

3. Give the points where the following functions are continuous.

   (a) $f(x) = \dfrac{x+1}{x^2+4x+3}$

   (b) $f(x) = \dfrac{x}{x^2+1}$

   (c) $f(x) = \sqrt{2x-3} + x^2$

   (d) $f(x) = \begin{cases} x^2+1 & \text{if } x \le 0 \\ x+1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \ge 2 \end{cases}$

4. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$

   (a) Find all numbers c such that $\lim_{x\to 1} f(x)$ exists.

   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

5. Find parameters a and b so that the following function is continuous

$$f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \le -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \le x \end{cases}$$

6. Suppose that

$$f(x) = \begin{cases} \dfrac{x-6}{|x-6|} & \text{if } x \ne 6 \\ 1 & \text{if } x = 6 \end{cases}$$

   (a) Determine the points at which the function f is not continuous.

(b) Is f continuous on the interval $[6, \infty)$?

(c) Is f continuous on the interval $(-\infty, 6]$?

7. State the Intermediate Value Theorem.

8. Using the Intermediate Value Theorem, find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

9. Let $f(x) = \dfrac{e^x}{e^x - 2}$.

   (a) Show that $f(0) < 1 < f(\ln 4)$.

   (b) Can you use the Intermediate Value Theorem to conclude that there is a solution of $f(x) = 1$ on the interval $[0, \ln(4)]$?

   (c) Can you find a solution to $f(x) = 1$?

§§2.1-2 Derivatives and tangent lines revisited

1. Comprehension check:

   (a) What is the definition of the derivative $f'(a)$ at a point a?

   (b) What is the geometric meaning of the derivative $f'(a)$ at a point a?

   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$?

2. Find the specified derivative for each of the following using the limit definition of derivative.

   (a) If $f(x) = 1/x$, find $f'(2)$.

   (b) If $f(x) = x^2$, find $f'(s)$.

   (c) If $f(x) = x^3$, find $f'(-2)$.

3. The point $P = (4, 2)$ lies on the curve $y = \sqrt{x}$.

   (a) If Q is the point $(x, \sqrt{x})$, find a formula for the slope of the secant line PQ.

   (b) Using your formula from part (a) and a calculator, find the slope of the secant line PQ for the following values of x (do not round until you get to the final answer):

   $$3.9,\ 3.98,\ 3.999,\ 4.1,\ 4.05,\ \text{and } 4.002$$

   TI-8x Calculator Tip: Enter the formula under `y=` and then use `Table`.

   (c) Using the results of part (b), guess the value of the slope of the tangent line to the curve at $P = (4, 2)$.

   (d) Verify that your guess is correct by computing the derivative from the definition.

   (e) Using the slope from part (d), find the equation of the tangent line to the curve at $P = (4, 2)$. Hint: You may check your work by graphing the function and the tangent line on the same set of axes. This is a good use of your calculator or Desmos.

4. (a) Find a function f and a number a so that the following limit represents a derivative $f'(a)$.

   $$\lim_{h \to 0} \frac{(-2+h)^3 + 8}{h}$$

   (b) Using your function f, set $h = 1/2$ and draw the graph of f and the secant line whose slope is given by $\frac{(-2+h)^3+8}{h}$.

5. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

6. Let
$$g(t) = \begin{cases} at + b & \text{if } t \leq 1 \\ t^2 & \text{if } t > 1 \end{cases}.$$
Find all values of a and b so that g is differentiable at $t = 1$.

7. Let $f(x) = e^x$ and compute the slopes of the secant lines which pass through $(0, f(0))$ and $(h, f(h))$ for $h = 0.1,\ 0.01,\ 0.001$. Next, find the slopes of the secant lines which pass through $(-h, f(-h))$ and $(h, f(h))$ for the same values of h.

   (a) Can you guess a value for the derivative $f'(0)$?

   (b) Which approach for computing the slope of the tangent line to the graph of f at 0 appears to be the better method? Explain why you prefer your answer.

8. Find A so that the limit
$$\lim_{x\to 2} \frac{x^2 + 2x - A}{(x - 2)}$$
is finite. Give the value of the limit.

§2.3 Interpretations of the derivative (part 1)

1. Let V be the volume of a spherical balloon in cubic centimeters and r is its radius.

   (a) What rate of change does the derivative dV/dr represent? What units should we use to express this rate of change?

   (b) Let t be time measured in seconds. What rate of change does dV/dt represent? What units should we use to express this rate of change?

2. Let F be temperature measured in degrees Fahrenheit and C be the temperature measured on the Celsius scale. Find the value of dF/dC. What is the value of dC/dF?

3. Suppose that the tangent line to f at $x = 3$ is $y = -3x + 3$. Is the rate of change of f at $x = 3$ positive or negative? Is f positive or negative at $x = 3$?

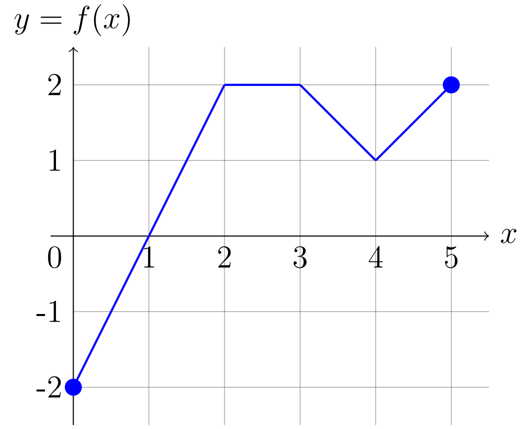
§2.3 Interpretations of the derivative (part 2)

1. Suppose N is the number of people in the United States who travel by car to another state for a vacation this year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain your answer.

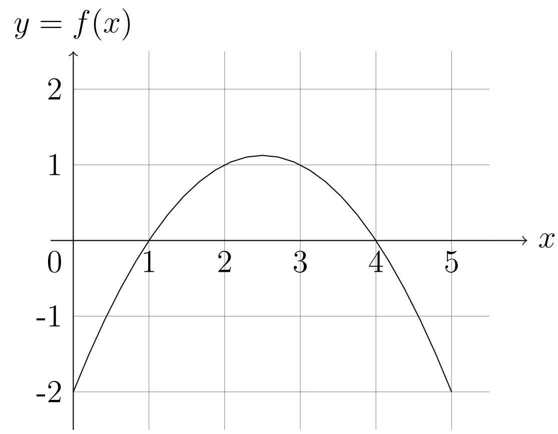
2. Consider the graph of f on the interval $[0, 5]$,

   (a) For which values of x is the function differentiable?

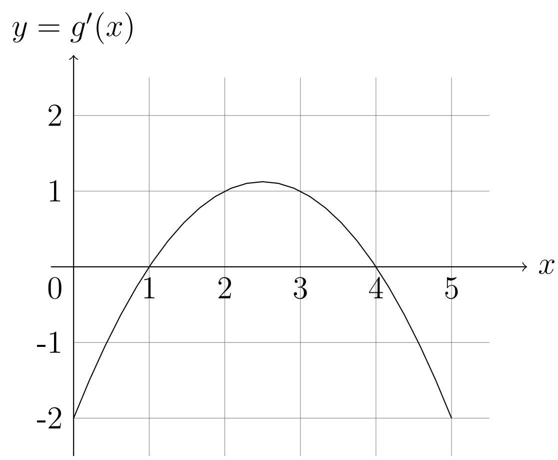
   (b) Sketch the graph of f'. Your graph should make clear where the derivative is undefined.



3. The graph of f is shown at the right. What can you say about f'? Can you find more than one possible answer?



4. The graph of g' is shown at the right. Sketch the graph of g. Can you find more than one possible answer?



5. The position of a particle P at time t is $p(t)$ units to the right of the origin and $p'(t) = 4 - t$. When is P moving to the right?

§2.4 The arithmetic of derivatives, §2.6 Using the arithmetic of derivatives

For these problems, use the following differentiation rules that we have established in earlier sections. Assume c is a constant.

$$\frac{d}{dx}c = 0, \qquad \frac{d}{dx}x = 1, \qquad \frac{d}{dx}x^2 = 2x, \qquad \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}, \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

1. Suppose that $f(1) = 2$, $f'(1) = 3$, $g(1) = 5$, and $g'(1) = 7$. Find the following.
   (a) $(2f + g)'(1)$
   (b) $(fg)'(1)$
   (c) $(f^2)'(1)$
   (d) $\left(\frac{1}{g}\right)'(1)$
   (e) $\left(\frac{f+g}{g}\right)'(1)$

2. Compute the following derivatives.
   (a) $\frac{d}{dx}(2x^2 + 3x)$
   (b) $\frac{d}{dx}\frac{x+1}{x-1}$
   (c) $\frac{d}{dx}((x+2)(3x^2 - \frac{1}{x}))$
   (d) $\frac{d}{dx}\frac{1}{x^2+4}$

3. Find the tangent line to $y = \frac{3x+1}{2x-2}$ at $x = 2$.

4. Suppose that the tangent line to f at $x = -1$ is $y = 3x - 1$,
   (a) Find $f(-1)$ and $f'(-1)$.
   (b) Let $g(x) = f(x)/x$ and find $g'(-1)$ and $g(-1)$.
   (c) Find the tangent line to the graph of g at $x = -1$.

5. There are (at least) two ways to compute the derivative

$$\frac{d}{dx}(2x+1)(3x-2)$$

   Compute the derivative by both of the following methods.
   (a) Use the product rule.
   (b) Multiply out the expression and then use the rule for the derivative of a sum.

   Check that both methods give the same result. Which method do you prefer?

6. Compute the derivative

$$\frac{d}{dx}\frac{x^3-1}{x^3+1}$$

using two different approaches.

(a) Use the quotient rule.

(b) Write the function to be differentiated as $(x^3-1)(x^3+1)^{-1}$ and use the product rule and reciprocal rule for derivatives.

Check that both methods give the same result. Which method do you prefer?

7. (a) Find the derivative of x^3 by writing $x^3 = x \cdot x^2$ and using the product rule.

(b) Find the derivative of x^4 by writing $x^4 = x^2 \cdot x^2$ and using the product rule. Can you think of another approach that will find this derivative?

(c) Compute the derivatives x^n for $n = 5, 6, 7$. Can you guess a formula for the derivative of x^n for $n = 1, 2, 3, \ldots$.

Review for Exam 1

The good news is you have already started to prepare for exam 1. Your solutions to WeBWorK, written assignments and worksheets give you a wealth of material to study. As you review your work, be sure to ask about problems that you find challenging and to study your lectures and/or textbook in order to The most important step in reviewing is to review your solutions to WeBWorK, written assignments and worksheets.

The following problems are provided for additional practice. Each was taken from exams given in previous semesters.

1. Find the inverse function to $f(x) = \dfrac{x+5}{2x-3}$. Give the domain and range of f and the inverse function f^{-1}.

2. The position function for a particle moving along a line is given by $s(t) = t^3 + 1$. Find the average velocity of the particle on the interval $[2, 5]$.

3. For each of the following limits, give the value of the limit if it a finite number or $\pm\infty$ or explain why it does not exist.

   (a) $\displaystyle\lim_{x\to 2} \frac{x+1}{x^2+2x}$.

   (b) $\displaystyle\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$.

   (c) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$.

   (d) $\displaystyle\lim_{x\to 2} \frac{x-3}{x-2}$.

   (e) $\displaystyle\lim_{x\to 2} \frac{x-3}{(x-2)^2}$.

4. Consider the piecewise defined function

$$f(x) = \begin{cases} 5 & \text{if } 0 \le x < 1 \\ ax + 3 & \text{if } 1 \le x < 2 \\ x^2 - 2x + b & \text{if } 2 \le x \le 3 \end{cases}$$

   where a and b are constants. Find the values of a and b for which $f(x)$ is continuous on $[0, 3]$.

5. (a) State the Intermediate Value Theorem.

   (b) Let $f(x) = x^5 - x^4 + 2x^2 - 4$. Use the Intermediate Value Theorem to show there must exist a solution to $f(x) = 4$ in the interval $[1, 2]$.

6. (a) State the formal definition of the derivative of a function $f(x)$ at the point $x = a$. *Hint:* Your definition should involve a limit.

(b) **Using the formal definition of the derivative and the limit laws,** find the derivative of the function $f(x) = x^2 + x - 1$ at $x = 1$.

7. Find $f(3)$ and $f'(3)$, assuming that the tangent line to $y = f(x)$ at $x = 3$ has equation $y = 4x - 3$.

§2.7 Exponential functions

1. Recall that the logarithm base a, $\log_a$, is the inverse function to the function a^x. Find the following logarithms.

   (a) $\log_3(9)$ (b) $\log_9(3)$ (c) $\log_{1/2}(4)$ (d) $\log_4(1/2)$

2. (a) Use a calculator to compute the quotient $\dfrac{a^h - 1}{h}$ for several different values of h with $a = 2.7$. Estimate the limit $C(a) = \lim_{h\to 0} \dfrac{a^h - 1}{h}$ for $a = 2.7$.

   (b) Use a calculator to compute the quotient $\dfrac{a^h - 1}{h}$ for several different values of h with $a = 2.8$. Estimate the limit $C(a) = \lim_{h\to 0} \dfrac{a^h - 1}{h}$ for $a = 2.8$.

   (c) According to the text, the number e is the number for which $C(e) = 1$. Based on your answer to part (a) and (b) what can you say about e.

3. Compute the derivatives of the following functions

   (a) xe^x

   (b) $(x^2 + 1)e^x$

   (c) $\dfrac{e^x + 1}{e^x - 1}$

   (d) $(e^x + x)(xe^x + 1)$

4. In §2.9, we will learn the chain rule and use it to compute the derivative of e^{ax}. For now, we can compute the derivative of $e^{2x} = e^x \cdot e^x$ using the product rule.

   (a) Use the product rule to find the derivatives of e^{2x} and e^{2x+1}.

   (b) Use the quotient rule or the reciprocal rule to find the derivative of e^{-x}.

5. (a) Find the tangent line to e^x at $x = 0$.

   (b) Use the tangent line at $x = 0$ to find an approximate value of $e = e^1$.

   (c) Another approach to approximating e is to use the tangent line at $x = 0$ to find an approximate value for $e^{0.1}$, write $e^1 = (e^{0.1})^{10}$ and then use our approximate value for $e^{0.1}$.

   Comment: While computing a 10th power is tedious, this approach gives an approximate value for e which only depends on knowing how to add and multiply.

   (d) Which method gives a better approximate value for e?

6. Suppose we find a very generous bank that will pay us 100% interest, compounded annually.

(a) If we deposit \$1 in the bank. How much we will have after one year?

(b) Now suppose that the interest compounded monthly. This means that after one month, we receive $1/12$ of the annual interest. This interest is added to our account and thus we earn interest on a little more than \$1 during the second month.

    How much will we have at the end of the year?

(c) What happens if the interest is compounded daily?

(d) What happens if the interest is compounded hourly?

§2.8 Trigonometric functions

1. Compute the derivatives below.

   (a) $a(x) = x\cos(x)$

   (b) $b(t) = \cos(t)\tan(t)$

   (c) $c(r) = \sec(r)\tan(r)$

   (d) $d(x) = e^x \sec(x)$

2. Let O be the center of a circle whose circumference is 48 centimeters. Let P and Q be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments OQ and OP. Express your answer in radians.

   Find the distance between P and Q.

3. For each of these statements, explain why it is true or give an example showing it is false.

   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.

   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \dfrac{2}{3}$, then the hypotenuse of the triangle must have length 3.

4. Use the basic trig limits to find the $\lim_{x\to 0} \dfrac{1-\cos(x)}{x^2}$.

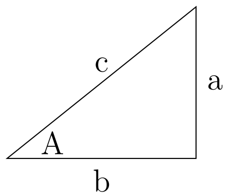
5. Use the derivatives of sin and cos and the quotient rule to verify the derivative of $\tan(x)$.

6. Consider the right triangle pictured. The angle labeled A is changing and the hypotenuse is c is fixed.

   (a) Find the rate of change of the length of side a with respect to the angle A.

   (b) Find the rate of change of the length of side b with respect to the angle A.

   (c) Find the rate of change of length of the side c with respect to the angle A.



7. Now consider the triangle from the previous problem, but suppose that the side a is kept constant, while the sides b, c and the angle A are varying.

   (a) Find the rate of change the length of side b with respect to the angle A

(b) Find the rate of change of the length of side c with respect to the angle A.

8. Let $f(t) = t + 2\cos(t)$.

   (a) Find all values of t where the tangent line to f at the point $(t, f(t))$ is horizontal.

   (b) What are the largest and smallest values for the slope of a tangent line to the graph of f?

§2.9 The chain rule

1. Let $f(x) = ax + b$ and $g(x) = cx + d$. What is the slope of $f \circ g$? Can you use the chain rule to answer this question?

2. Find the derivatives of the following functions.

   (a) $a(x) = e^{x^2}$

   (b) $b(r) = \sin(\cos^2(r))$

   (c) $c(x) = \sin(x^2)e^{5x^3}$

   (d) $d(t) = \sin(\sin(\sin(t)))$

   (e) $e(x) = \dfrac{x + \sin^2(3x)}{x^2 - 4\cos(x^2)}$

   (f) $f(t) = \sin(2t)\cos(3t)\sin(5t)$

3. If we have the graph of a function f, then the graph of $g(x) = f(2x)$ is obtained by compressing the graph by a factor of 2 along the x-axis. What is the relation between the slope of the tangent lines to f and the slopes of the tangent lines to g? You might want to consider the graphs of $\sin(x)$ and $\sin(2x)$.

   Verify your answer using the chain rule.

4. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2e^{3x}$, $x = 2$

   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

5. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

   Find: $h'(-1), h'(3)$ and $k'(2)$.

6. Differentiate both sides of the identity $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar result?

7. (a) Let $f(x) = e^{2x}$ and show that $f'(x) = 2f(x)$.

   (b) Can you find a solution of $g'(x) = g(x)$. Can you find more than one solution?

   (c) Suppose that $h'(x) = h(x)$. What can you say about the derivative of $h(x)/e^x$?

8. If $x = e^{f(x)}$ and f is a differentiable function, differentiate both sides to obtain an equation involving f'. Can you find a simple expression for f'?

§2.10 The natural logarithm

Our textbook uses $\log(x)$ for the natural logarithm $\log_e(x)$. In the US it is more common to use $\ln(x)$ for the function $\log_e(x)$. Most calculators use $\ln(x)$ for the natural logarithm and $\log(x)$ for the common logarithm or log base 10, $\log_{10}(x)$. In this course we will use log or ln for the natural logarithm. For the logarithm for any base a besides $a = e$, we will write the base as a subscript as in $\log_a(x)$.


MIRIFICI

Logarithmorum

Canonis descriptio,

Ejusque usus, in utraque *Trigonometria; ut etiam in* omni Logistica Mathematica, *Amplissimi, Facillimi, & expeditissimi explicatio.*

Authore ac Inventore, IOANNE NEPERO, Barone Merchistonii, *&c. Scoto.*

EDINBVRGI. Ex officinâ ANDREÆ HART *Bibliopôlæ,* CIↃ. DC. XIV.

Some history: While several people independently developed the idea of the logarithm, the most influential of these was John Napier through his 1614 book *Mirifici Logarithmorum Canonis Descriptio*. Napier developed the theory of logarithms to allow faster calculation, and it was so successful that within a few decades his logarithms had spread across the globe due to the promotion of Henry Briggs and Edward Wright in England, Bonaventura Cavalieri in Italy, Johannes Kepler in Germany, and Xue Fengzuo in China. Through reprintings of the book *Shu Li Ching Yün*, originally published in Beijing by Emperor K'ang Hsi, Napier's theory of logarithms reached Japanese mathematicians in the early 1700's.

1. (a) Find the tangent line to $y = e^x$ at $x = 0$.
   (b) Sketch the graph of $f(x) = e^x$ and the inverse function to f.
   (c) What is the tangent line to the inverse function at $x = 1$?

2. Find the derivatives of the following functions.
   (a) $a(x) = \ln(1 + \sin^2(x))$
   (b) $b(x) = \ln(xe^{-2x})$
   (c) $c(x) = x\ln(x) - x$
   (d) $d(x) = \dfrac{\sin(2\ln(x))}{\cos(e^{2x})}$

3. Find an equation of the tangent line to the curve at the given point.
   (a) $y = \ln(x^2 - 3x + 1), \quad (3, 0)$
   (b) $y = x^2\ln(x), \quad (1, 0)$

4. Use that $a^b = e^{b\ln(a)}$ to find the derivative of 2^x.

5. If $f(x) = a^x$ and $f' = 3f$, find a.

6. Solve the equation $e^{x^2+3x} = e^4$.

7. Recall the properties of the logarithm

   - $\ln(ab) = \ln(a) + \ln(b)$
   - $\ln(a^r) = r\ln(a)$
   - $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$

   (a) Use these properties to simplify the following expression

   $$F(x) = \ln((x+1)^2(x^3+x))$$

   Then compute the derivative.

   (b) Check your answer by computing the derivative of F without first simplifying.

   (c) Which method do you find easier?

8. If a function F involves a product or quotient, then we can use the logarithm to help find the derivative. Simplify $\ln(F)$ as in the previous problem and then compute the derivative of $\ln(F)$. For example, if $F(x) = \dfrac{a(x)b(x)}{c(x)}$, then

   $$\ln(F(x)) = \ln\left(\frac{a(x)b(x)}{c(x)}\right) = \ln(a(x)) + \ln(b(x)) - \ln(c(x)).$$

   Hence, the derivative is

   $$\frac{F'(x)}{F(x)} = \frac{a'(x)}{a(x)} + \frac{b'(x)}{b(x)} - \frac{c'(x)}{c(x)}.$$

   Lastly, solve for $F'(x)$.

   $$F'(x) = F(x)\left(\frac{a'(x)}{a(x)} + \frac{b'(x)}{b(x)} - \frac{c'(x)}{c(x)}.\right)$$

   This method is called *logarithmic differentiation*.

   Use logarithmic differentiation to find the derivatives of the following functions.

   (a) $F(x) = \dfrac{3+x}{3-x}$.

   (b) $F(x) = \dfrac{(x+1)(x+2)^2(x+3)^3}{(x-1)}$.

§2.11 Implicit differentiation

1. Find the points on the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ where $x = 3$. Find the tangent lines to the ellipse at these points.

   Make a sketch to check your answer.

   See `https://www.desmos.com/calculator/oyhtjnc9mg`

2. Suppose $x(y) = e^{y^2}$, find dx/dy and dy/dx

3. Find the derivative of y with respect to x:

   (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.

   (b) $e^y \sin(x) = x + xy$.

   (c) $\cos(xy) = 1 + \sin(y)$.

4. (CLP §2.9 Exercise 14.) If $x^2 + (y+1)e^y = 5$, find the tangent lines to this curve at the points where $y = 0$.

5. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

6. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

7. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

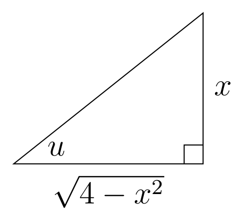
8. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

§2.12 Inverse trigonometric functions

1. (a) Let $f(x) = \sin(x)$ for x in the domain $[-\pi/2, \pi/2]$. Sketch the graph of f and the inverse function f^{-1}. Use the graph of f^{-1} to sketch the graph of the derivative of the inverse function, $(f^{-1})'$. Since f^{-1} is the function arcsin, you may check if your derivative looks right.
   (b) Let $g(x) = \sin(x)$ for x in the domain $[\pi/2, 3\pi/2]$. Sketch the graph of g and the inverse function g^{-1}. Use the graph of g^{-1} to sketch the graph of the derivative of the inverse function, $(g^{-1})'$.
   (c) From your answers to parts a) and b), what do you think the relation between the two derivatives is? Are they equal?
   (d) Find the derivative of the function g^{-1}.

2. A ladder of length 6 meters leans against a tall wall so that the base of the ladder is 2 meters from the base of the wall.
   (a) Make a sketch based on the given information.
   (b) How high up the wall does the ladder reach?
   (c) Find the angle that the ladder forms with the wall.

3. Use the right triangle below to express the following expressions in terms of x.
   (a) $\sin(u)$
   (b) $\cos(u)$
   (c) $\tan(u)$



4. Compute the following expressions. Be careful!!
   (a) $\arcsin(\sin(23\pi/6))$ note that $\arcsin(\sin(x)) = x$ only for x in the restricted domain $-\pi/2 \leq x \leq \pi/2$.
   (b) $\cos(\arcsin(1/2))$
   (c) $\arccos(\sin(\pi/3))$
   (d) $\cos(\arcsin(2))$

5. Compute the derivatives of the following functions and give the domain of the derivatives.
   (a) $f(x) = \arcsin(x/4)$

(b) $g(t) = \dfrac{\arccos(t)}{t^2 - 1}$

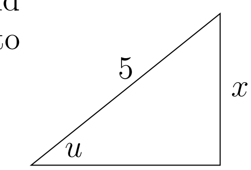
(c) $h(y) = \tan^{-1}(1 + y^2)$

(d) $k(x) = \operatorname{arcsec}(x^2 - 1)$

6. Compute the derivative of $f(x) = \arctan(x) + \arctan(1/x)$ and simplify the resulting expression. Can you explain why the derivative simplifies to a simple expression?

7. Determine all points on the curve $y = \arcsin(x)$ where the tangent line is parallel to the line $y = 2x + 9$.

8. The hypotenuse of the triangle is fixed at 5. Find the rate of change of the angle u with respect to the side labelled x.



§2.13 The mean value theorem

1. (a) State the Mean Value Theorem.

   (b) Does the function $f(x) = x^3$ satisfy the hypotheses of the mean value theorem on the interval $[1, 3]$.

   (c) For f on the interval $[1, 3]$, find all values of c as guaranteed in the mean value theorem.

2. Consider $f(x) = |x|$ on the interval $[-1, 1]$. Is there a value c so that $f'(c) = \dfrac{f(1) - f(-1)}{1 - -1}$? Explain why the mean value theorem does not apply.

3. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.

4. Let f be differentiable for all x. Suppose we know that $1 \leq f(2) \leq 4$ and that $2 \leq f'(x) \leq 5$ for all x. What are the largest and smallest possible values for $f(6)$? What about $f(-1)$?

5. If f is differentiable and $f'(x) = \dfrac{\sin(x)}{1 + x^4}$, is $f(1)$ greater than, equal to or less than $f(0)$?

6. Use the Mean Value Theorem to show that $\sin(x) \leq x$ for $x \geq 0$. What can you say for $x \leq 0$?

7. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 158 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why did she deserve the ticket?

§2.14 Higher order derivatives

1. Compute the specified derivatives.

   (a) $\dfrac{d^2}{dt^2}\dfrac{1}{1+3t^2}$

   (b) $f^{(3)}(r)$ if $f(r) = e^{r^2}$

   (c) $g''(s)$ where $g(s) = \sin(s^2+1)$

2. Let $f(x) = e^{2x}$ and find $f^{(n)}$ for $n = 1, 2, 3, 4$. Can you guess a formula for $f^{(n)}$?

3. Let $g(x) = \sin(x)$. What is $g^{(101)}(x)$? Find $g^{(2023)}(x)$.

4. Recall that the expression $n! = n(n-1)\ldots 2\cdot 1$ is the product of the first n whole numbers with the convention that $0! = 1$.

   Compute $n!$ for $n = 0, 1, 2, 3, 4, 5, 6$.

5. Find $\dfrac{d^n}{dx^n}x^n$ for $n = 1, 2, 3, 4$. Can you guess a formula for general n?

6. (a) Let $f(x) = e^x$ and find a quadratic polynomial $p(x)$ so that $p^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2$. (The zeroth derivative $f^{(0)}$ is just the function f.)

   (b) With p as in part a), find $p(0.1)$. If we use $p(0.1)$ as an approximate value for $e^{0.1}$, what is the error $|e^{0.1} - p(0.1)|$?

   (c) Since $e = (e^{0.1})^{10}$, we may use $p(0.1)^{10}$ to approximate e. We may also use $p(1)$ to approximate e. Which is the better approximation to e?

   (d) Can you suggest a way to find an even better approximation to e?

§3.1 Velocity and acceleration

1. Given that $G(t) = 4t^2 - 3t + 42$ find the instantaneous rate of change when $t = 3$.

2. A particle moves along a line so that its position at time t is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin. Recall that *speed* is given by the absolute value of velocity.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

3. Suppose that an object is shot into the air vertically with an initial velocity v_0 and initial height s_0, with acceleration due to gravity denoted by g. Let $s(t)$ denote the height of the object after t time units.

   (a) Explain why $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$ (this is sometimes called "Galileo's formula").

   (b) If time is measured in seconds and distance in meters, what are the units for s_0, v_0, and g?

4. An object is thrown upward so that its height at time t seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time t.

5. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s^2.

6. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s^2.

7. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

   (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.

   (b) Is the particle moving towards or away from the origin at π seconds? Explain.

(c) Now, find the velocity of the particle at time $t = \dfrac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?

(d) Is the particle speeding up at $\frac{\pi}{2}$ seconds?

§3.4.1-3 Approximating at a point by linear and quadratic functions

1. (a) Use the linear approximation of $\sqrt{x}$ at $a = 16$ to estimate $\sqrt{18}$.

   (b) Find a decimal approximation to $\sqrt{18}$ using a calculator.

   (c) Compute both the absolute error and the percentage error if we use the linear approximation to approximate $\sqrt{18}$.

   Hint: If A is an approximate value to a number N, then absolute error is $|A - N|$ and the percentage error is $100 \cdot |A - N|/|N|$.

2. (a) Find the linear and quadratic approximation to $f(x) = x^3$ at $x = 3$.

   (b) Use these approximations to find approximate values for $(3.02)^3$ and $(2.97)^3$ and compare to the exact values.

3. (a) Find the linear and quadratic approximation to $f(x) = \tan(x)$ at $x = \pi/4$.

   (b) Use these approximations to find an approximate values for $\tan(44°)$.

4. (a) Find the linear and quadratic approximations to $f(x) = \sqrt[3]{x}$ at $x = 8$.

   (b) Use these approximations to find approximate values for $\sqrt[3]{8.5}$.

5. (a) A sphere of radius r has volume $\dfrac{4}{3}\pi r^3$. Find the linear approximation to this function at $r = 2$ m.

   (b) Suppose we want to paint a sphere of radius 2 m with a coat of paint that is 0.5 cm thick. Use a linear approximation to approximate the volume of paint we need to do the job.

   Hint: Be careful with the units. Recall that cm is an abbreviation for centimeter, m is an abbreviation for meter and 100 cm equals 1 m.

6. Your physics professor tells you that you can replace $\sin(\theta)$ with θ when θ is close to zero. Explain why this is reasonable.

7. Suppose that $y = y(x)$ is a differentiable function which is defined near $x = 2$, satisfies $y(2) = -1$ and
$$x^2 + 3xy^2 + y^3 = 9.$$
   Use the linear approximation to the change in y to approximate the value of $y(1.91)$.

8. (a) Find the linear and quadratic approximations to $\cos(x)$ at $x = 0$.

   (b) Use the results for part a) to find approximate values to $\cos(0.1)$. Find the error for each of these approximations.

(c) Recall the double angle formula for cosine in the form

$$\cos(2x) = 2\cos^2(x) - 1.$$

Use the following procedure to approximate $\cos(0.1)$.

i. Use the quadratic approximation find an approximate value for $\cos(0.05)$.
ii. Use the double angle formula and our approximate value for $\cos(0.05)$ to find an approximate value for $\cos(0.1)$.

(d) Compare the approximate values for $\cos(0.1)$ in part b) and part c). Which is the best approximation?

Review for Exam 2

1. Find the slope of the tangent line to the graph of $f(x) = x^2e^{2x}$ at $x = 2$.

2. Find the equation of the tangent line to $f(x) = \sqrt{3x+1}$ at $x = 5$.

3. (a) Find all points on $y^2 + 2xy + 2x^2 = 8$ with $x = 2$.

   (b) Find $\frac{dy}{dx}$ for $y^2 + 2xy + 2x^2 = 8$.

   (c) Find the equation of the tangent line to $y^2 + 2xy + 2x^2 = 8$ at each point with $x = 2$.

4. Let $f(x) = x^3 + 3x - 1$ and let g be the inverse of f. Use implicit differentiation to find an expression for g' in terms of $g(x)$.

   Find the tangent line to the graph of g when $x = 3$.

5. (a) State the Mean Value Theorem.

   (b) Suppose that $g(x)$ is differentiable for all x and that $-2 \leq g'(x) \leq 4$ for all x. Also assume that $g(1) = 3$. Find the largest possible value for $g(3)$.

6. A watermelon is dropped off a tall building so that its height in meters at time t in seconds if $h(t) = -4.9t^2 + 200$. Find the velocity when it hits the ground.

7. Find the linear and quadratic approximations to $\sqrt{x}$ at $x = 81$. Use the linear and quadratic approximations to find approximate values for $\sqrt{80}$ and $\sqrt{83}$.

§3.2 Related rates

1. Let a and b denote the length in meters of the two legs of a right triangle. At time $t = 0$, $a = 20$ and $b = 20$. If a is decreasing at a constant rate of 2 meters per second and b is increasing at a constant rate of 3 meters per second. Find the rate of change of the area of the triangle at time $t = 5$ seconds.

2. A spherical snow ball is melting. The rate of change of the surface area of the snow ball is constant and equal to -7 square centimeters per hour. Find the rate of change of the radius of the snow ball when $r = 5$ centimeters.

3. The height of a cylinder is a linear function of its radius (i.e. $h = ar + b$ for some a, b constants). The height increases twice as fast as the radius r and $\frac{dr}{dt}$ is constant. At time $t = 1$ seconds the radius is $r = 1$ feet, the height is $h = 3$ feet and the rate of change of the volume is 16π cubic feet/second.
   (a) Find an equation to relate the height and radius of the cylinder.
   (b) Express the volume as a function of the radius.
   (c) Find the rate of change of the radius when the radius is 4 feet.

4. A water tank is shaped like a cone with the vertex pointing down. The height of the tank is 5 meters and diameter of the base is 2 meters. At time $t = 0$ the tank is full and starts to be emptied. After 3 minutes the height of the water is 4 meters and it is decreasing at a rate of 0.5 meters per minute. At this instant, find the rate of change of the volume of the water in the tank. What are the units for your answer? Recall that the volume of a right-circular cone whose base has radius r and of height h is given by $V = \frac{1}{3}\pi r^2 h$.

5. A plane flies at an altitude of 5000 meters and a speed of 360 kilometers per hour. The plane is flying in a straight line and passes directly over an observer.
   (a) Sketch a diagram that summarizes the information in the problem.
   (b) Find the angle of elevation 2 minutes after the plane passes over the observer.
   (c) Find rate of change of the angle of elevation 2 minutes after the plane passes over the observer.

6. Let $f(x) = \dfrac{1}{1 + x^3}$ and $h(x) = \dfrac{1}{1 + f(x)}$
   (a) Find $f'(x)$.
   (b) Use the previous result to find $h'(x)$.
   (c) Let $x = x(t)$ be a function of time t with $x(1) = 1$ and set $F(t) = h(x(t))$. If $F'(1) = 18$, find $x'(1)$.

§3.3 Exponential growth and decay

1. Suppose that a population of bacteria triples every hour and starts with 400 bacteria.

   (a) Find an expression for the number n of bacteria after t hours.

   (b) Use this expression to estimate the rate of growth of the bacteria population after 2.5 hours.

   (c) How long does it take the population to double?

2. Suppose a population has a growth rate of 4% per year so that the population will increase from 100 to 104 in one year.

   (a) If the initial population is $P(0) = 100$, find the population $P(t)$ after t years.

   (b) Show that $P' = rP$ and give the value of r. Is r greater than, equal to or smaller than 0.04?

3. (a) The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

   (b) How long does it take the the mass to decrease to 1/3 of its value?

4. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.

$$\frac{dP}{dt} = KP$$

   where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$

   (a) Give the units of K.

   (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$. How many mosquitoes will there be after 90 days?

   (c) How long will it take for the population of mosquitoes to double?

5. A lucky colony of rabbits is brought to a large island where there are no predators and unlimited food. Under these conditions, they will reproduce at such a rate that the population doubles every 9 years. After 3 years, a team of scientists determines that there are 7000 rabbits on the island.

   (a) How many rabbits were brought to the island originally?

(b) How many rabbits will there be 12 years after their introduction to the island?

6. A cup of coffee at temperature 80°C is placed in a room where the temperature is 25°C. If the temperature of the coffee at 12 noon is 60°C and at 1pm it is 40°C, when was the coffee placed in the room?

7. Think about the other science courses you are currently taking (or have taken in the past). Identify three to five examples of problems from those courses that involve rates of change where the language of calculus might be useful.

§3.5.1-2 Maximum and minimum values

1. State the theorem which guarantees that a function has a maximum and minimum value (see Theorem 3.5.11).

2. For each of the functions determine the global maximum and minimum values, and determine if Theorem 3.5.11 applies.

   (a) $f(x) = 1 + x$ on the interval $[0, 1)$.

   (b) $g(x) = \dfrac{x}{x+2}$ on the interval $[1, \infty)$

   (c) $h(x) = \begin{cases} 1 - x, & 0 < x \leq 1 \\ 0, & 0 \end{cases}$ on the interval $[0, 1]$

   (d) $k(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ on the interval $(-\infty, \infty)$.

3. Use calculus to find the local extrema for the following functions. You may use a calculator or Desmos to check your answer.

   (a) $f(x) = xe^{-x}$

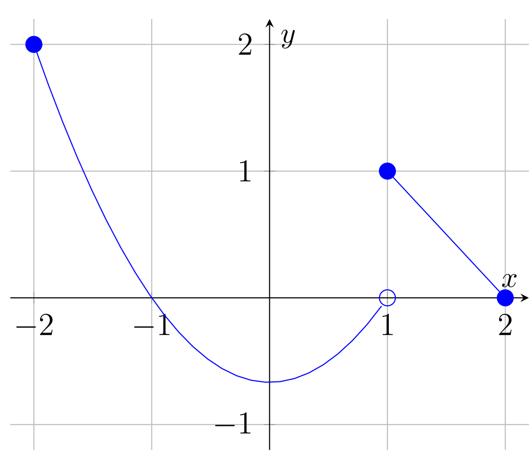
   (b) $g(x) = x^3 - 3x^2 - 9x$

   (c) $h(x) = x + 2\sin(x)$

4. (a) Sketch the graph of a function with domain the real line, a global maximum at 2 and no minimum value.

   (b) Sketch the graph of a function with domain the closed interval [0,5], with local maxima at $x = 2$ and $x = 4$, local minimum at $x = 3$, a global maximum at $x = 4$ and no global minimum.

5. Find the local extrema and the global extrema for the function given by the graph.



6. For each function, list the critical points and singular points and find the global extreme values. You may use a graph to check your answer.

   (a) $f(x) = 1 + x - x^2$ on the interval $[-1, 4]$.

   (b) $g(x) = \cos(x)\sin(x) - \cos(x)$ on the interval $[0, 3\pi/2]$.

   (c) $h(x) = (x - 2)^{2/3}$ on the interval $[0, 4]$.

7. Let $f(x) = x^4$.

   (a) Find the critical point.

   (b) Is the critical point a local minimum, local maximum, or neither?

   (c) Can you use the second derivative test (Theorem 3.5.5) to answer part b)?

§3.6.1-2, The first derivative and the graph

1. Determine the domain of each of the functions and give any horizontal and vertical asymptotes of the graph.

   (a) $f(x) = \dfrac{x^2 + 9}{x^2 - 3x - 10}$

   (b) $g(x) = \ln(x^2 - 9)$

   (c) $h(x) = \dfrac{x^2 - 9}{x^2 - 6x + 9}$

2. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.

      i. Find the critical number(s) of $f(x)$.

      ii. Find the intervals on which $f(x)$ is increasing or decreasing.

      iii. Find the local extrema of $f(x)$.

   (b) Repeat with the function $f(x) = \dfrac{x}{x^2 + 4}$ on $(-\infty, \infty)$.

3. Consider the function $f(x) = x^4 - 8x^3 + 5$.

   (a) Find the intervals on which the graph of $f(x)$ is increasing or decreasing.

   (b) Find the local extrema of f.

4. Consider the function $f(x) = \dfrac{x^2}{4 - x^2}$.

   (a) Find the intervals where the graph of f is increasing or decreasing.

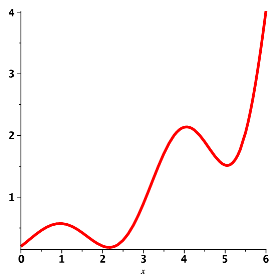
   (b) Find the horizontal asymptote(s), vertical asymptote(s) and y-intercept(s).

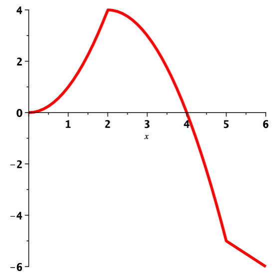
5. Sketch the graph of a function that is

   - continuous on $(-\infty, 2) \cup (2, 5) \cup (5, \infty)$
   - increasing on $(-\infty, 2)$
   - increasing on $(2, 5)$
   - decreasing on $(5, \infty)$
   - has vertical asymptotes at $x = 2$ and $x = 5$

§3.6.3 The second derivative and concavity

1. Comprehension Check:
   (a) Explain what the first derivative tell us about the graph of a function.
   (b) Explain what the second derivative f'' tells us about the graph of f.
2. Sketch a graph of a continuous function $f(x)$ with the following properties:
   - f is increasing on $(-\infty, -3) \cup (1, 7) \cup (7, \infty)$
   - f is decreasing on $(-3, 1)$
   - f is concave up on $(0, 3) \cup (7, \infty)$
   - f is concave down on $(-\infty, 0) \cup (3, 7)$
3. Consider the function $f(x) = e^{-x^2}$.
   (a) Find the intervals for which f is increasing or decreasing.
   (b) Find the intervals for which the graph of f is concave up or concave down.
   (c) Find the local extrema and inflection points for the graph.
   (d) Make a sketch of the graph of f that uses the above information.
4. Consider the function $f(x) = xe^x$.
   (a) Give the domain and any asymptotes.
   (b) Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
   (c) Find the local extrema of f.
   (d) Find the inflection points of $f(x)$.
   (e) Find the intervals of concavity of $f(x)$.
   (f) Make a sketch which includes the above information.
5. Consider the function $f(x) = 2x + \sin(x)$ on $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
   (a) Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
   (b) Find the local extrema of f.
   (c) Find the inflection points of $f(x)$.
   (d) Find the intervals of concavity of $f(x)$.
   (e) Make a sketch which includes the above information.
6. Below are the graphs of two functions.





(a) Find the intervals where each function is increasing and decreasing respectively.

(b) Find the intervals of concavity for each function.

(c) For each function, identify all local extrema and inflection points on the interval (0,6).

§3.5.3 Optimization

1. Find the dimensions of x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating x and y?

   (b) Find a formula for the area in terms of x alone.

   (c) Solve the optimization problem.

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

4. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

5. Find the dimensions x and y of the rectangle inscribed in a circle of radius r that maximizes the quantity xy^2. For this problem, assume that r is fixed and your will answer will depend on the value of r.

6. Suppose that f is a function on an open interval $I = (a, b)$ and c is in I. Suppose that f is continuous at c, $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$. Is $f(c)$ an global minimum value for f on I? Justify your answer.

7. Find the point(s) on the hyperbola $y = \dfrac{16}{x}$ that is (are) closest to $(0, 0)$. Be sure to clearly state what function you choose to minimize or maximize and why.

8. Consider a can in the shape of a right circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter, and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72 π cubic centimeters, and whose cost is as small as possible.

   (a) Find a function $f(r)$ which gives the cost of the can in terms of radius r. Be sure to specify the domain.

   (b) Give the radius and height of the can with least cost.

   (c) Explain how you known you have found the can of least cost.

9. A box is to have a square base, no top, and a volume of 32 cubic centimeters.

   (a) Make a sketch and label the sides.

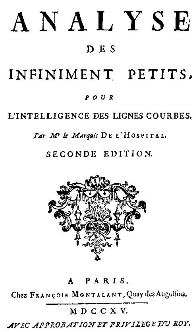
(b) Write a function for the total area of the sides of the box in terms of one of the dimensions from your sketch.

    Is it easier to express the area in terms of the side length of the square base or the height of the box?

(c) What are the dimensions of the box with the smallest possible total surface area?

(d) Explain how you know you have found the smallest possible surface area?

§3.7 L'Hôpital's rule



ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis De L'Hospital.
SECONDE EDITION.
A PARIS,
Chez François Montalant, Quay des Augustins,
MDCCXV.
AVEC APPROBATION ET PRIVILEGE DU ROY.

Some history: L'Hospital's Rule was probably not discovered by l'Hospital!

> The Marquis de l'Hôpital was a French nobleman and amateur mathematician who [wanted to learn] calculus. [He] employed Johann Bernoulli to supply him with tracts on various aspects of the subject, as well as to provide him with any new mathematical discoveries of note. In a sense, it appears that l'Hôpital bought the rights to Bernoulli's mathematical research [and published it under his own name as] *Analyse de infiniment petits.* — William Dunham

1. Carefully state l'Hôpital's Rule for the indeterminate form ∞/∞.

2. (a) Find the limit $\displaystyle\lim_{x\to 0}\frac{x^2+2x+1}{x^2+x+1}$.
   (b) If we try to use l'Hôpital's rule to evaluate the limit in part a), we will need to consider the limit $\displaystyle\lim_{x\to 0}\frac{2x+2}{2x+1}$. Find this limit.
   (c) Are your answers in a) and b), the same? Explain why or why not.
   (d) Can you find two functions f and g so that
   $$\lim_{x\to 0}\frac{f(x)}{g(x)}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{f'(x)}{g'(x)}=42$$

3. Compute the following limits. Use l'Hôpital's Rule where appropriate, but first check that no easier method will solve the problem.
   (a) $\displaystyle\lim_{x\to 1}\frac{x^9-1}{x^5-1}$
   (b) $\displaystyle\lim_{x\to 1}\frac{x^2+2x-2}{x^2-2x+2}$
   (c) $\displaystyle\lim_{x\to\infty}\frac{3x+2\sqrt{x}}{1-x}$
   (d) $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{\tan(5x)}$
   (e) $\displaystyle\lim_{x\to\infty}\frac{\ln(1+x)}{x}$
   (f) $\displaystyle\lim_{x\to\infty}\sqrt{x}e^{-x}$
   (g) $\displaystyle\lim_{x\to-\infty}\frac{2x-5}{|3x+2|}$
   (h) $\displaystyle\lim_{x\to 0}(1+\sin(x))^{1/x}$

4. Consider the limit $\displaystyle\lim_{x\to\infty}\frac{5x^2+\sin x}{3x^2+\cos x}$.

(a) Attempt to use l'Hôpital's rule to evaluate the limit. What happens?

(b) Find the limit.

5. Consider the limit $\lim_{x \to \infty} x^3 e^{-x}$.

   (a) Write $x^3 e^{-x} = x^3/e^x$ and attempt to use l'Hôpital's rule to evaluate the limit.

   (b) Write $x^3 e^{-x} = e^{-x}/x^{-3}$ and attempt to use l'Hôpital's rule to evaluate the limit.

   (c) Which approach is more useful?

6. Suppose we are very lucky and find a bank that will pay 100% interest, compounded n times per year.

   (a) If we deposit one dollar in the bank, how much will we have after 1 year? Your answer will depend on n.

   (b) What happens as n becomes large?

7. Suppose we know:

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

   Which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

   (a) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$

   (b) $\lim_{x \to a} \dfrac{p(x)}{f(x)}$

   (c) $\lim_{x \to a} p(x)q(x)$

   (d) $\lim_{x \to a} \dfrac{p(x)}{q(x)}$

8. Find the value A for which we can use l'Hôpital's rule to evaluate the limit

$$\lim_{x \to 2} \frac{x^2 + Ax - 2}{x - 2}.$$

   For this value of A, give the value of the limit.

§4.1 Anti-derivatives

Some history: Antiderivatives, areas, and distances are fundamental in physics and mathematics. Pierre-Simon Laplace published *Mécanique Céleste* in five volumes between 1798 and 1825, and this is generally considered the next major work on gravitational mathematics and celestial mechanics after Newton's *Principia*. Eager to have a version in English, the Society for the Diffusion of Useful Knowledge commissioned a translated and expanded version from Mary Somerville, a famous Scottish mathematician and astronomer, which was published in 1831 under the title *The Mechanism of the Heavens*. In part due to the phenomenal success of her translation and extensions of the work of Laplace, in 1835 Somerville was one of the first two women (jointly with Caroline Herschel) to become a member of the Royal Astronomical Society.

1. Comprehension Check:
   - (a) If F is an antiderivative of a continuous function f, is F continuous? What if f is not continuous?
   - (b) Let $g(x) = \dfrac{x^3}{3} + 1$. Find $g'(x)$. Now give two antiderivatives of $g'(x)$.
   - (c) Let $h(x) = x^2 + 1$, and let $H(x)$ be any antiderivative of h. What is $H'(x)$?

2. Find the general antiderivatives of the following functions
   - (a) $f(x) = x^2 - 3x + 2 - \frac{5}{x}$.
   - (b) $g(x) = (x-5)^2$.
   - (c) $h(x) = 2\sin(x) - \sec^2(x)$.
   - (d) $f(x) = 1 + 2\sin(x) + \dfrac{3}{\sqrt{x}}$.

3. Find f given that
$$f'(x) = \sin(x) - \sec(x)\tan(x), \qquad f(\pi) = 1.$$

4. Find g given that
$$g''(t) = -9.8, \qquad g'(0) = 1, \qquad g(0) = 2.$$
   On the surface of the earth, the acceleration of an object due to gravity is approximately $-9.8\,\mathrm{m/s^2}$. What situation could we describe using the function g? Be sure to specify what g and its first two derivatives represent.

5. A stone is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
   - (a) Find the distance of the stone above the ground at time t.

(b) How long does it take the stone to reach the ground?

(c) With what velocity does it strike the ground?

(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

6. Two balls are thrown upward from the edge of a 147 meters above the ground. The first is thrown with a speed of 19.6 meters/second and the second is thrown a second later with a speed of 14.7 meters/second. Do the balls ever pass each other?

7. A high-speed bullet train accelerates and decelerates at the rate of 2 meters/second^2. Its maximum cruising speed is 180 kilometers per hour.

   (a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?

   (b) Suppose that the train starts from rest and must come to a complete stop, all within 15 minutes. What is the maximum distance it can travel under these conditions?

   (c) Find the minimum time that the train takes to travel between two consecutive stations that are 90 kilometers apart.

   (d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

II-§1.1.1-6 Definition of the integral

The following summation formulas will be useful below.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Students may also find the following Riemann sum calculator in Desmos helpful:

`https://www.desmos.com/calculator/cxsfmpvf69`

1. Write each of following in summation notation:
   (a) $1+2+3+4+5+6+7+8+9+10$
   (b) $2+4+6+8+10+12+14$
   (c) $2+4+8+16+32+64+128$.
2. Compute $\displaystyle\sum_{i=1}^{4}\left(\sum_{j=1}^{3}(i+j)\right)$.
3. Use summation formula above to find
   (a) $\displaystyle\sum_{k=1}^{20}(3k+2)$
   (b) $\displaystyle\sum_{k=1}^{11}(2k^2-5)$
   (c) $\displaystyle\sum_{k=1}^{n}(2k-1)$ Hint: The answer simplifies nicely.
4. Let $f(x) = \frac{1}{x}$. Divide the interval $[1, 3]$ into five subintervals of equal length. The left and right Riemann sums are defined in Definition 1.1.11 of §1.1.4 of CLP-2.
   (a) Find the right Riemann sum R_5 to approximate the area $\int_1^3 f(x)dx$. Is R_5 larger or smaller than the true area?
   (b) Find the left Riemann sum L_5 to approximate the area $\int_1^3 f(x)dx$. Is R_5 larger or smaller than the true area?
5. Let $f(x) = \sqrt{1-x^2}$. Divide the interval $[0, 1]$ into four equal subintervals and compute the left and right Riemann sums L_4 and R_4. Again L_4 uses the left endpoint of each subinterval to find the height of the subrectangles and R_4 uses

the right endpoint of each subinterval to find the height for the approximating rectangle.

Is R_4 larger or smaller than the true area?

Is L_4 larger or smaller than the true area?

What can you conclude about the value π?

6. Let $f(x) = x^2$.

   (a) If we divide the interval $[0, 2]$ into n sub intervals of equal length, how long is each interval?

   (b) Write a sum R_n which approximates the area under the graph of f where we on the interval $[0, 2]$. We find R_n by finding the area of n rectangles where each rectangle has its base on one of the equal subintervals of $[0, 2]$ and whose heights is given by the value of f at the right endpoint of the subinterval.

   (c) Use one of the formulae for the sums of powers of whole numbers to find a closed form expression for R_n.

   (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.

7. Use the summation formulae at the beginning of this worksheet to give the value of the following sums.

   (a) $\displaystyle\sum_{k=1}^{20}(2k^2 + 3)$

   (b) $\displaystyle\sum_{k=11}^{20}(3k + 2)$

8. Recognize the sum as a Riemann sum and express the limit as an integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}.$$

9. Let $f(x) = x$ and consider the partition $P = \{x_0, x_1, \ldots, x_n\}$ which divides the interval $[1, 3]$ into n subintervals of equal length.

   (a) Find a formula for x_k in terms of k and n.

   (b) We form a rectangle whose width is $\Delta x = (x_k - x_{k-1})$ and whose height is $f(x_k)$. Give the area of the rectangle.

(c) Choose the sample points to be the right endpoint of each subinterval. Form the Riemann sum, and use the formula for sums of powers to simplify the Riemann sum.

(d) Take the limit as n tends to infinity to find the area of the region under $f(x)$ for $1 \leq x \leq 3$.

(e) Find the area above using geometry to check your answer.

II-§1.2 Properties of the definite integral (part 1)

1. Suppose

$$\int_0^1 f(x)\,dx = 2, \text{ and } \int_1^2 f(x)\,dx = 3,$$

$$\int_0^1 g(x)\,dx = -1, \text{ and } \int_0^2 g(x)\,dx = 4$$

.

Compute the following using the properties of definite integrals:

(a) $\displaystyle\int_1^2 g(x)\,dx$

(b) $\displaystyle\int_0^2 [2f(x) - 3g(x)]\,dx$

(c) $\displaystyle\int_1^1 g(x)\,dx$

(d) $\displaystyle\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx$

(e) $\displaystyle\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx$

2. Simplify

$$\int_a^b f(t)\,dt + \int_b^c f(u)\,du + \int_c^a f(v)\,dv.$$

3. Suppose that we know $\int_0^x f(t)\,dt = \sin(x)$, find the following integrals.

(a) $\displaystyle\int_0^{\pi} f(t)\,dt$

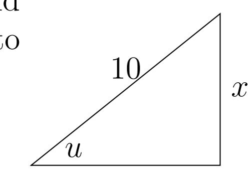
(b) $\displaystyle\int_{-\pi/2}^{0} f(t)\,dt$

(c) $\displaystyle\int_{\pi/2}^{\pi} f(t)\,dt$

(d) $\displaystyle\int_{-\pi}^{\pi} f(t)\,dt$

Review for Exam 3

1. The hypotenuse of the triangle is fixed at 10. Find the rate of change of the angle u with respect to the side labelled x.



2. The side length l of a square is increasing. When $l = 5$ the derivative $\dfrac{dl}{dt} = 6$. Let A be the area of the square and find $\dfrac{dA}{dt}$ when $l = 5$.

3. The half-life of a radioactive substance is 20 years. A sample of the substance has a mass of 24 grams. In how many years will the substance have a mass of 4 grams?

4. We have 240 meters of fencing and form a rectangular pen that is divided in two by a fence parallel to two of the sides. Find the area and dimensions of the pen which encloses the largest area.

5. Find f if $f''(x) = \sin x + \cos x$, $f'(0) = 8$, and $f(0) = 2$.

II-§1.2 Properties of the definite integral (part 2)

1. (a) Use geometry to find the integral $\int_0^a s\,ds$.

   (b) Use your answer to part a) and properties of the integral to find:

      i. $\int_2^5 t\,dt$

      ii. $\int_4^1 u\,du$

2. Suppose that f is a continuous function and $6 \leq f(x) \leq 7$ for x in the interval $[3, 9]$.

   (a) Find the largest and smallest possible values for $\int_3^9 f(x)\,dx$.

   (b) Find the largest and smallest possible values for $\int_8^4 f(x)\,dx$.

3. Find $\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$.

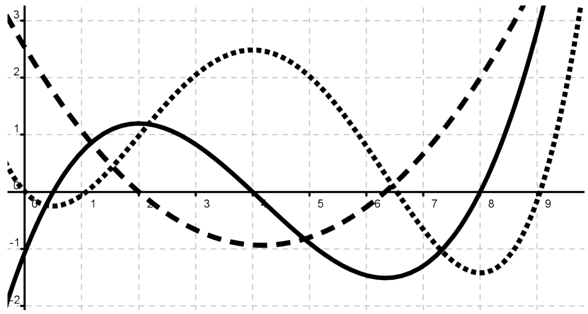
   Hint: The integral represents the area of region that can be divided into triangles and squares.

4. Find the largest and smallest values of $f(x) = \cos(x)$ on the interval $[-\pi/4, \pi/4]$. Use this information to find numbers m and M so that

$$m \leq \int_{-\pi/4}^{\pi/4} \cos(x)\,dx \leq M.$$

II-1.3 The fundamental theorem of Calculus

1. Below is pictured the graph of the function $f(x)$, its derivative $f'(x)$, and the integral $\int_0^x f(t)\,dt$. Identify $f(x), f'(x)$ and $\int_0^x f(t)\,dt$ and explain your reasoning.

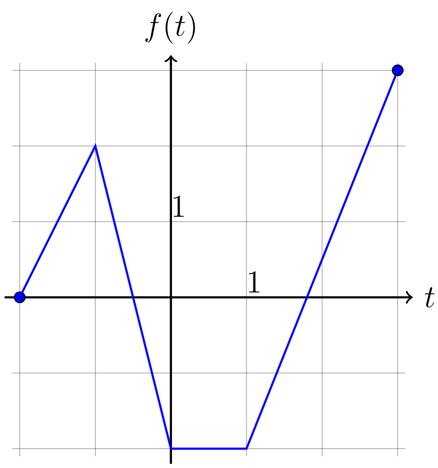


2. Let $g(x) = \int_{-2}^x f(t)\,dt$ where f is the function whose graph is shown below.

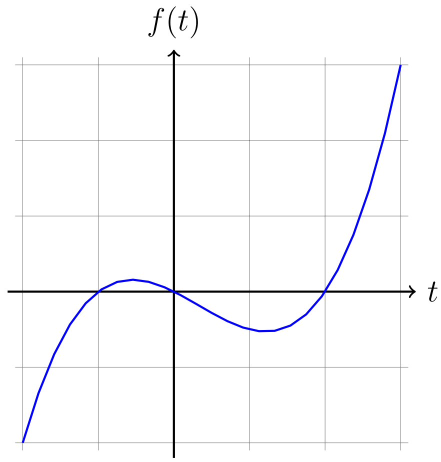
   (a) Evaluate $g(-1)$, $g(0)$, $g(1)$, $g(2)$, and $g(3)$.

   (b) On what interval is g increasing? Why?

   (c) Where does g have a maximum value? Why?



3. Let $g(x) = \int_{-2}^x f(t)\,dt$ where f is the function whose graph is shown below. Where is $g(x)$ increasing and decreasing? Explain your answer.



4. Let $F(x) = \int_2^x e^{t^2}\, dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2.

5. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the following functions:

   (a) $g(x) = \displaystyle\int_1^x \left(2 + t^4\right)^5\, dt$

   (b) $F(x) = \displaystyle\int_x^4 \cos\left(t^5\right)\, dt$

   (c) $h(x) = \displaystyle\int_0^{x^2} \sqrt[3]{1 + r^3}\, dr$

   (d) $y(x) = \displaystyle\int_{1/x^2}^0 \sin^3(t)\, dt$

6. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the following integrals or explain why the theorem does not apply:

   (a) $\displaystyle\int_{-2}^5 6x\, dx$

   (b) $\displaystyle\int_{-2}^7 \frac{1}{x^5}\, dx$

   (c) $\displaystyle\int_{-1}^1 e^{u+1}\, du$

   (d) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sin(2x)}{\sin(x)}\, dx$

7. Find the following integrals.

   (a) $\displaystyle\int_1^4 x^2 + x\, dx$

   (b) $\displaystyle\int_1^e \frac{dx}{x}$

   (c) $\displaystyle\int_0^{999\pi} \sin(x)\, dx$

   (d) $\displaystyle\int_0^1 e^x + x^e\, dx$

II-§1.4 Substitution

1. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int \frac{4}{(1+2x)^3}\,dx$

   (b) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

   (c) $\displaystyle\int \cos^4(\theta)\sin(\theta)\,d\theta$

   (d) $\displaystyle\int \sec^3(x)\tan(x)\,dx$

   (e) $\displaystyle\int e^x \sin\left(e^x\right)\,dx$

   (f) $\displaystyle\int \frac{2x+3}{x^2+3x}\,dx$

2. Evaluate the following definite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right)\,dx$

   (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x^2}}\,dx$

   (d) $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

3. If f is continuous on $(-\infty,\infty)$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx\,.$$

   For the case where $f(x)\geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

4. Assume f is a continuous function.

   (a) If $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 x\cdot f(x^2)\,dx$.

   (b) If $\displaystyle\int_0^u f(x)\,dx = 1+e^{u^2}$ for all real numbers u, find $\displaystyle\int_0^2 f(2x)\,dx$.

5. Recall that we can express any exponential function in terms of the exponential base e by writing writing $b^x = e^{x\ln(b)}$? Use this to find the indefinite integral $\displaystyle\int b^x\,dx$.

6. Which integral should be evaluated using substitution? Evaluate both integrals:

(a) $\displaystyle\int \frac{9\,dx}{1+x^2}$

(b) $\displaystyle\int \frac{x\,dx}{1+9x^2}$

7. Find a so that if $x = au$, then $16 + x^2 = 16(1+u^2)$. Use this to find the anti-derivative
$$\int \frac{1}{16+x^2}\,dx.$$

8. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

(a) $\displaystyle\int \frac{\cos(\ln(t))\,dt}{t}$

(b) $\displaystyle\int \frac{x\,dx}{\sqrt{7-x^2}}$

(c) $\displaystyle\int \frac{dt}{4t^2+9}$

(d) $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$

(e) $\displaystyle\int e^{9-2x}\,dx$

II-§1.5 Area between curves

1. Find the area of the region between the graphs of $y = x^2$ and $y = x^3$.
2. Find the area of the regions enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x + \frac{3}{4}$ in two ways.
   (a) By writing an integral in x.
   (b) Solve each equation to express x in terms of y and write an integral with respect to y.
3. Find the area enclosed by the curves $y = x^2$ and $x = y^2$.
4. Find the area of the entire region enclosed by the graphs of $y = x + 1$ and $y = x^3 + x^2 - x + 1$.
5. Find the area of one of the regions enclosed by $y = \cos(x)$ and $y = \sin(x)$.
6. (a) Find the area enclosed by the curve x^{-1-a}, $x = 1$, $x = 2$, and $y = 0$.
   (b) Find the limit as a tends to zero of the area you found in part a). Can you give a reason why the value of the limit makes sense?

Review for Exam 4

Exam 4 will be cumulative, with an emphasis on the material covered since Exam 3. The questions below cover the new material since Exam 3. Students should review all material covered in the course to prepare for Exam 4.

1. Assuming that $\int_0^5 f(x)\,dx = 5$ and $\int_0^5 g(x)\,dx = 12$, find

$$\int_0^5 \left(3f(x) - \frac{1}{3}g(x)\right)\,dx$$

2. Let $f(x) = \int_0^{x^2} \ln(1+t)\,dt$. What is $f'(x)$?

3. Compute the following antiderivatives.

   (a) $\int \frac{2e^x}{(5e^x+3)^4}\,dx$

   (b) $\int x\sqrt{x+1}\,dx$

   (c) $\int \frac{\cos(x)}{(1+\sin(x))^3}\,dx$

4. Find the area between the curves $y = x + 1$ and $y = x^3 + x^2 - x + 1$.

January 22, 2024