MA 114 — Calculus II Spring 2014
Exam 2 Key March 11, 2014

Name: ______________________________

Section: ______________________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:**
  Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:**
  Show all your work on the page of the problem. Show all your work. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	**C**	D	E
2	A	B	**C**	D	E
3	**A**	B	C	D	E
4	A	B	C	**D**	E

Exam Scores

Question	Score	Total
MC		20
5		15
6		18
7		18
8		14
9		15
Total		100

Unsupported answers for the free response questions may not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. Which of the following is the correct form for the partial fraction decomposition of $\dfrac{4x^2+5}{(x-3)^2(2x+3)}$?

A. $\dfrac{Ax+B}{(x-3)^2} + \dfrac{C}{2x+3}$

B. $\dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{Cx+D}{(2x+3)}$

C. $\dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(2x+3)}$

D. $\dfrac{Ax+B}{(x-3)} + \dfrac{Cx+D}{(2x+3)}$

E. None of the above

2. Which of the following integrals represents the area of the surface obtained by revolving the curve $y = \cos(x)$ between $x = 0$ and $x = \pi/2$ about the x-axis?

A. $\int_0^{\pi/2} \pi \cos^2 x \, dx.$

B. $\int_0^{\pi/2} \sqrt{1+\sin^2 x} \, dx.$

C. $\int_0^{\pi/2} 2\pi \cos(x)\sqrt{1+\sin^2 x} \, dx.$

D. $\int_0^{\pi/2} \sqrt{1+\cos^2 x} \, dx.$

E. $\int_0^{\pi/4} 2\pi \sin(x)\sqrt{1+\cos^2 x} \, dx.$

Record the correct answer to the following problems on the front page of this exam.

3. Suppose that $y(t)$ satisfying the initial value problem

$$\begin{aligned} y' &= 3(y-2) \\ y(0) &= 4 \end{aligned}$$

Then:

A. $\lim_{t\to+\infty} y(t) = +\infty$

B. $\lim_{t\to+\infty} y(t) = 0$

C. $\lim_{t\to+\infty} y(t) = 4$

D. $\lim_{t\to+\infty} y(t) = 3$

E. $\lim_{t\to+\infty} y(t) = 5$

4. Which of the following is the Taylor polynomial of order 4 (expanding about $x = 0$) for the function $f(x) = \frac{1}{2}(e^x + e^{-x})$?

A. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

B. $1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{24}$

C. $1 + x + \dfrac{x^3}{6}$

D. $1 + \dfrac{x^2}{2} + \dfrac{x^4}{24}$

E. $1 - \dfrac{x^2}{2} - \dfrac{x^4}{24}$

5. (15 points)

(a) (8 points) Find the partial fraction decomposition of

$$\frac{10}{(x-1)(x^2+9)}$$

Solution:

$$\begin{aligned}\frac{10}{(x-1)(x^2+9)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+9}\\ 10 &= A(x^2+9) + (Bx+C)(x-1)\\ 10 &= (A+B)x^2 + (C-B)x + (9A-C)\end{aligned}$$

so that

$$\begin{aligned}A+B &= 0\\ C-B &= 0\\ 9A-C &= 10\end{aligned}$$

The solution is $A = 1$, $B = -1$, $C = -1$ so that

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{(x-1)} + \frac{-x-1}{x^2+9}$$

Suggested scoring: Score 2 points for correct PFD, 1 point for placing correctly over a common denominator, 3 points for a correct solution for A, B, C, and 2 points for final answer.

(b) (7 points) Evaluate the integral

$$\int \frac{2x-5}{x^2+1}\,dx$$

Solution:

$$\begin{aligned}\int \frac{2x-5}{x^2+1}\,dx &= \int \frac{2x}{x^2+1}\,dx + \int \frac{-5}{x^2+1}\,dx\\ &= \ln(x^2+1) + (-5)\arctan(x) + C\end{aligned}$$

Suggested scoring: 3 points for a correct decomposition of the integral into "sub-problems" and 2 points each for the correct evaluation of the integrals.

6. (18 points) Determine whether each of the following improper integrals is convergent or divergent. If the integral converges, evaluate the integral.

(a) (9 points) $\int_0^\infty \frac{1}{1+x}\,dx$

Solution: First, by definition

$$\int_0^\infty \frac{1}{1+x}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{1+x}\,dx.$$

Next, by the substitution $u = 1+x$ or other correct method, compute

$$\begin{aligned}\int_0^R \frac{1}{1+x}\,dx &= \int_1^{1+R} \frac{1}{u}\,du \\ &= \ln(1+R)\end{aligned}$$

Conclude that the integral diverges because

$$\lim_{R\to\infty} \ln(1+R) = \infty.$$

Suggested scoring: 3 points for the correct definition of the improper integral as a limit, 3 points for correct evaluation of the integral from 0 to R and 3 points for a correct argument that the integral diverges.

(b) (9 points) $\int_1^2 \frac{1}{x(\ln x)^2}\,dx$

Solution: The integrand is discontinuous at $x = 1$ so the integral, if it exists, is given by

$$\lim_{a\to 1^+}\int_a^2 \frac{dx}{x(\ln x)^2}.$$

To evaluate this integral make the substitution $u = \ln x$ so that

$$\begin{aligned}\int_a^2 \frac{dx}{x(\ln x)^2} &= \int_{\ln a}^{\ln 2} \frac{du}{u^2} \\ &= \left[-\frac{1}{u}\right]\Bigg|_{\ln a}^{\ln 2} \\ &= \frac{1}{\ln 2} - \frac{1}{\ln(a)}\end{aligned}$$

Since $\ln(a) \to 0$ as $a \to 1$, this integral *diverges*. *Suggested Scoring*: 3 points for correctly identifying the discontinuity and setting up the improper integral as a limit, 3 points for evaluating $\int_a^2 dx/(x(\ln(x))^2)$, and 3 points for a correct argument that the integral diverges.

7. (18 points) Find the area of the surface of revolution obtained by revolving

$$y = x^3$$

on the interval $[0, 2]$ about the x-axis.

Solution: With $f(x) = x^3$ and $f'(x) = 3x^2$, the integral for surface area is given by

$$\begin{aligned} S &= \int_0^2 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^2 2\pi x^3 \cdot \sqrt{1 + 9x^4}\, dx \end{aligned}$$

This integral can be evaluated by the substitution $u = 1 + 9x^4$, $du = 36x^3\, dx$ to get

$$\begin{aligned} S &= \int_1^{145} \frac{1}{18}\pi\sqrt{u}\, du \\ &= \frac{1}{27}\pi\left(145\sqrt{145} - 1\right) \\ &\simeq 203.043 \end{aligned}$$

Suggested scoring: 6 points for correctly setting up the integral. Stating the general formula is not mandatory. 6 points for making the u-substitution, and 6 points for a correct evaluation of the integral.

8. (14 points) Solve the initial value problem

$$\begin{aligned} y' &= (2x-1)(y-2) \\ y(2) &= 4 \end{aligned}$$

Solution: This is a separable equation. Separate variables:

$$\begin{aligned} \frac{dy}{dx} &= (2x-1)(y-2) \\ \frac{dy}{y-2} &= (2x-1)\,dx \\ \int \frac{dy}{y-2} &= \int (2x-1)\ dx \\ \ln|y-2| &= x^2 - x + C \\ y-2 &= Ce^{x^2-x} \end{aligned}$$

Now use the initial condition and the above formula to compute

$$\begin{aligned} 2 &= Ce^2 \\ C &= 2e^{-2} \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= 2 + 2e^{-2}e^{x^2-x} \\ &= 2 + 2e^{x^2-x-2} \end{aligned}$$

Suggested scoring: 4 points for correctly separating variables, 4 points for integrating to obtain $\ln|y-2| = x^2+x+C$, 2 points to solve for y, 2 points for correctly evaluating C, and 2 points for the final answer.

9. (15 points)

(a) (5 points) State Simpson's rule for approximating $\int_a^b f(x)\,dx$ using $N = 4$ intervals of size Δx.

Solution: With $x_i = a + i\Delta x$ and $y_i = f(x_i)$,

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

Suggested scoring:1 point for correct factor $\Delta x/3$, 2 points for the correct coefficients on $y_0, \cdots y_4$, 2 points for explaining that $y_i = f(x_i)$ and $x_i = a + i\Delta x$ or otherwise making these identifications correctly.

(b) (10 points) The identity

$$\int_1^2 \frac{1}{x}\,dx = \ln(2)$$

gives us a way to compute $\ln(2)$ using Simpson's rule. How many intervals N would be required to use Simpson's rule in order to compute $\ln(2)$ with an error of no more than 5×10^{-5}? Recall that the error estimate for Simpson's rule applied to $\int_a^b f(x)\,dx$ is

$$\text{Error}(S_N) \leq \frac{K_4(b-a)^5}{180N^4}$$

where K_4 is a constant greater than or equal to $f^{(4)}(x)$, the fourth derivative of f, on $[a, b]$.

Solution: Since

$$f(x) = \frac{1}{x}$$

it is easy to see that

$$f^{(4)}(x) = -24/x^5.$$

Hence, we can choose $K = 24$ in the error estimate. We wish to choose N so that

$$\frac{24}{180N^4} \leq 5 \times 10^{-5}$$

or

$$N^4 \geq \left(\frac{1}{5} \times 10^5\right) \times \frac{24}{180}.$$

Hence

$$N^4 \geq \frac{8000}{3}.$$

The choice $N = 8$ is the smallest possible N since

$$8^4 = 4096, \quad 7^4 = 2401, \quad 8000/3 = 2666.\overline{6}.$$

Suggested scoring: 3 points for correctly determining K, 4 points for correctly setting up the inequality for N, 3 points for correctly concluding that $N = 8$ is the least value which will produce the desired error.