Exam 2

Name: ________________________________ Section: ________________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Give the first four terms of the sequence $\{a_1, a_2, a_3, a_4\}$ defined by

$$a_n = \frac{\cos(n\pi)}{n!}.$$

A. $\{-1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}\}$
B. $\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}\}$
C. $\{0, \frac{2}{\sqrt{2}}, \frac{3\sqrt{2}}{4}, \frac{-\sqrt{2}}{3}\}$
D. $\{-1, 0, \frac{-1}{6}, 0\}$
E. $\{-\pi, \frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{24}\}$

2. (5 points) Find the limit of the **sequence** $\{a_1, a_2, a_3, \ldots\}$ defined by

$$a_n = \frac{10n}{\sqrt{4n^2+9}}.$$

A. 2
B. 0
C. ∞
D. 5
E. $\frac{5}{2}$

3. (5 points) Find the sum of the series $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{n+4} - \frac{1}{n+5}\right)$.

A. $\frac{1}{4}$
B. $\frac{1}{7}$
C. $\frac{1}{20}$
D. $\frac{1}{56}$
E. This series diverges

4. (5 points) Use the formula for the sum of a geometric series to find the sum

$$\frac{5^3}{7} + \frac{5^4}{7^2} + \frac{5^5}{7^3} + \frac{5^6}{7^4} + \cdots$$

A. $\frac{7}{2}$

B. $\frac{5}{7}$

C. $\frac{432}{7}$

D. $\frac{125}{2}$

E. This series diverges.

5. (5 points) Does the series $\sum_{n=6}^{\infty} \frac{\sqrt{n}}{n^2 - 30}$ converge or diverge?

A. Converges by the ratio test since $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{(n+1)^2 - 30} \cdot \frac{n^2 - 30}{\sqrt{n}} \right| < 1$.

B. Converges because $\lim_{n\to\infty} \frac{\sqrt{n}}{n^2 - 30} = 0$.

C. Diverges by the limit comparison test to $\sum_{n=6}^{\infty} \frac{1}{\sqrt{n}}$.

D. Diverges by the ratio test since $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{(n+1)^2 - 30} \cdot \frac{n^2 - 30}{\sqrt{n}} \right| = 1$.

E. Converges by the limit comparison test to $\sum_{n=6}^{\infty} \frac{1}{n^{3/2}}$**.**

6. (5 points) What would you compare $\sum_{n=1}^{\infty} \frac{2^n + 7}{5^n - 3}$ to for a conclusive limit comparison test?

   A. $\sum_{n=1}^{\infty} \left(\frac{9}{2}\right)^n$

   B. $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$

   C. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

   D. $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$

   E. The limit comparison test can't be used to understand convergence for this series.

7. (5 points) Does the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$ converge absolutely, converge conditionally, or diverge?

   A. Converges absolutely because $\left|\frac{\cos(n)}{n^3}\right| \leq \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

   B. Converges conditionally by the alternating series test.

   C. Diverges since $\lim_{n\to\infty} \cos(n)$ does not exist.

   D. Converges absolutely since $\lim_{n\to\infty} \frac{\cos(n)}{n^3} = 0$.

   E. Converges conditionally since $-1 \leq \cos(n) \leq 1$.

8. (5 points) Find the smallest value of N so that S_N approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ to within an error of at most .21.

   A. $N = 16$

   B. $N = 18$

   C. $N = 20$

   D. $N = 22$

   E. $N = 24$

9. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} n!(x+5)^n$?

A. $(-\infty, \infty)$

B. $\{-5\}$

C. $[-13, 3]$

D. $\{0\}$

E. $[-13, 3)$

10. (5 points) Which power series represents the function $f(x) = x^2 \cos(3x^4)$ on the interval $(-\infty, \infty)$?

A. $\sum_{n=0}^{\infty} \frac{(-3)^n x^{4n+2}}{(2n)!}$

B. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{6n}}{(2n)!}$

C. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{8n+2}}{(2n)!}$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+3}}{(2n+1)!}$

E. $\sum_{n=0}^{\infty} \frac{(-9)^n x^{16n+2}}{(2n)!}$

Free Response Questions

11. Decide if the series converges or diverges. Clearly state which test(s) are used, and show all steps.

 (a) (5 points) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

Solution: Converges by the ratio test since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \lim_{n\to\infty} \left|\frac{3}{n+1}\right| = 0 < 1.$$

 (b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n^3-1}$

Solution: Converges by the limit comparison test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series.

$$\lim_{n\to\infty} \frac{\frac{n}{4n^3-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{4n^3-1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{4n^3-1} = \frac{1}{4},$$

and $0 < \frac{1}{4} < \infty$.

12. Are the series absolutely convergent, conditionally convergent or divergent? Clearly state which test(s) are used, and show all steps.

(a) (6 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$

Solution: The series converges conditionally.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n-1} \right| = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

which diverges by a direct comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ (p-series with $p = 1$).

We have,

$$\lim_{n\to\infty} \frac{1}{2n-1} = 0 \text{ and } \frac{1}{2(n+1)-1} < \frac{1}{2n-1} \text{ (decreasing)}$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the alernating series test.

(b) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7}$

Solution: The series diverges by the divergence test since $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{7} \neq 0$.

13. (10 points) Find the interval of convergence for the series. Hint: Show clearly where you test the endpoints of your interval.

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{7^n}$$

Solution: Using the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+2)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n(x+2)^n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{x+2}{7}\right| = \left|\frac{x+2}{7}\right|$$

Thus we need $-1 < \frac{x+2}{7} < 1$, or $-7 < x+2 < 7$, so $-9 < x < 5$.

If $x = -9$ then $\sum_{n=1}^{\infty} \frac{n(-7)^n}{7^n} = \sum_{n=1}^{\infty} n(-1)^n$ diverges by the divergence test.

If $x = 5$ then $\sum_{n=1}^{\infty} \frac{n7^n}{7^n} = \sum_{n=1}^{\infty} n$ diverges by the divergence test.

Thus the interval of convergence is $(-9, 5)$.

14. (a) (4 points) Write a Taylor series centered at $x = 0$ for the function $f(x) = \dfrac{2}{1-x^7}$.

Solution:

$$\frac{2}{1-x^7} = 2\sum_{n=0}^{\infty}(x^7)^n = \sum_{n=0}^{\infty} 2x^{7n}$$

(b) (6 points) Use your answer in (a) to help find the series for $g(x) = \dfrac{x^6}{(1-x^7)^2}$ centered at $x = 0$. Hint: First compute $f'(x)$.

Solution: $f(x) = 2(1-x^7)^{-1}$, $f'(x) = -2(1-x^7)^{-2}(-7x^6) = \dfrac{14x^6}{(1-x^7)^2}$.

$$f(x) = \sum_{n=0}^{\infty} 2x^{7n}$$

$$f'(x) = \frac{14x^6}{1-x^7} = \sum_{n=1}^{\infty} 2(7n)x^{7n-1}$$

Thus,

$$\frac{x^6}{(1-x^7)^2} = \frac{1}{14}\sum_{n=1}^{\infty} 14nx^{7n-1} = \sum_{n=1}^{\infty} nx^{7n-1}$$

15. (a) (4 points) Write the Maclaurin series, i.e., the Taylor Series centered at $x = 0$, for

$$f(x) = x^3 e^{2x}$$

.

Solution:

$$f(x) = x^3 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = x^3 \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+3}}{n!}$$

(b) (6 points) Write the first four terms of the Taylor series centered at $x = 7$ for

$$f(x) = \ln x$$

.

Solution:

$$\begin{aligned} f(x) &= \ln x & f(7) &= \ln 7 \\ f'(x) &= \frac{1}{x} & f'(7) &= \frac{1}{7} \\ f''(x) &= \frac{-1}{x^2} & f''(7) &= -\frac{1}{49} \\ f'''(x) &= \frac{2}{x^3} & f'''(7) &= \frac{2}{343} \end{aligned}$$

The first four terms are

$$\ln 7 + \frac{1}{7}(x-7) - \frac{1}{49(2)}(x-7)^2 + \frac{2}{343(6)}(x-7)^3.$$